

## THE SOLVING OF A NON-LINEAR PROBLEM OF THERMAL TRANSFER BY A PARTICULAR ANALYTICAL METHOD

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### ABSTRACT

This paper deals with the distribution of temperature in an ingot heated inside a pit furnace. In order to determine the non-stationary thermal field we resorted to a linear discretizing of the thermo-physical parameters along time intervals. For each interval we considered a polynomial interpretation of temperature, imposing besides the connection conditions we also considered the checking of the Galerkin integral.

The analytical results we obtained were compared to the experimental measurements given in the reference literature, which lead to satisfactory approximations.

### KEYWORDS:

ingot, non-stationary thermal field, pit furnace

### 1. INTRODUCTION

The cracking of alloy steel ingots during heating up, is caused by the thermal stress, whose value reaches the temperature limit before the core temperature exceeds the elasticity limit (500 – 550 °C). The choice of the proper hot working technology for these steel grades is first dependant on the heating conditions, decided upon according to their technological characteristics. The choice of the optimal heating pattern is most often done according to practical experience.

This paper is meant to introduce an analytical method enabling a most accurate estimation of the temperature field decided upon for the heating up of these ingots, thereby allowing considerations on the magnitude and distribution of thermal stress resulting from the heating process.

The method we suggest is a direct one and represents a powerful instrument for the study of linear and non-linear problems in mechanics and physics. As a result of this direct method we obtained an approximate analytical solution to the equation under consideration. In many cases, practical engineering considers such an analytical solution, even if approximate, preferable to the “exact”, numeric one. Finding a solution to the equation of thermal conduction is usually associated to major difficulties, particularly because of its non-linearity (material properties that are dependent on temperature) and also because of the limit conditions related to the radiation phenomenon. In this study we focused on the non-linear problems caused by the temperature variation of the material properties.

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The method we have introduced consists in sequentially turning thermal coefficients linear, in order to simplify the main equation. As a result of this approach, instead of having one non-linear differential equation with partial derivatives, we have an equivalent set of linear differential equations. In the case under consideration, this method gives satisfactory results for a wide range of temperatures

The authors consider that the problems related to the radiation conditions still persist (and they will be a topic of high interest for many researches) but, by this approach at least, and for certain temperature ranges, the results are acceptable.

The temperature values obtained by means of the method we introduced will be compared to the magnitudes obtained experimentally in Romania by Prof. Ilca Ioan Dr. es Sc. [1].

## 2. THE MATHEMATICAL MODEL

In order to derive the mathematical method appropriate to the ingot-heating phenomenon we have to take a few hypotheses into consideration. From the physical point of view, we are dealing with a rigid, isotropic body (the ingot), inside which a non-stationary phenomenon of thermal conduction takes place. From the geometrical standpoint, the ingot can be assimilated to a parallelepiped with a square basis and an infinite height.

Because of the cross-sectional symmetry and of the relative position of the ingot inside the pit-furnace (see fig. 1), the behavior of the system under consideration can be described by means of a parabolic differential equation simulating a mono-directional conduction phenomenon, and also with the help of an initial limit (frontier) condition given below [2]:

$$\rho(T) \cdot c_p(T) \cdot \frac{\partial T}{\partial t}(y,t) = \frac{\partial}{\partial y} \left[ \lambda(T) \cdot \frac{\partial T}{\partial y}(y,t) \right] \quad (1)$$

$$T(y,0) = T_0 \quad (2)$$

$$\lambda(T) \cdot \frac{\partial T}{\partial y}(0,t) = \varepsilon \cdot \sigma_0 \cdot [T_f^4(t) - T^4(0,t)] + \alpha \cdot [T_f(t) - T(0,t)] = q_1(t) \quad (3)$$

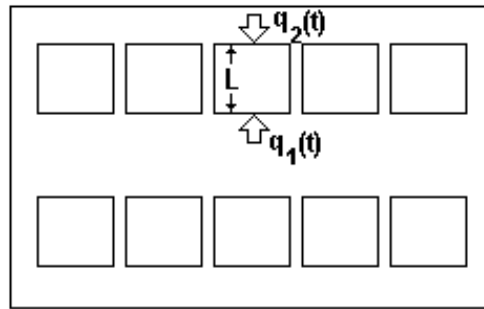
$$\lambda(T) \cdot \frac{\partial T}{\partial y}(L,t) = \varepsilon \cdot \sigma_0 \cdot [T_f^4(t) - T^4(L,t)] + \alpha \cdot [T_f(t) - T(L,t)] = q_2(t) \quad (4)$$

where:  $T = T(y,t)$  is the temperature distribution in the ingot cross section, dependant on width  $y$  and time  $t$ . The ingot lateral is marked  $L$ ;  $\rho$  is the specific mass,  $c_p$  is the specific heat, and  $\lambda$  represents the thermal conductivity of the steel under consideration, all of them depending on temperature. Moreover,  $\varepsilon$  is the emission factor of the ingot surface and  $\sigma_0 = 5,67 \cdot 10^{-8} \text{ W}/(\text{m}^2\text{K}^4)$  is the radiation constant of the perfectly black body, or the Stefan - Boltzmann constant. The initial temperature of the ingot is  $T_0=293\text{K}$ , and the temperature inside the pit furnace is marked  $T_f(t)$ . Finally,  $\alpha$  [ $\text{W}/(\text{m}^2\text{K})$ ] represents the mean coefficient of convection heat transfer at the surface of the ingot.

We looked for a solution to equation (1), of the form:

$$T(y,t) = a_1(t) + b(t) \cdot y + c(t) \cdot y^2 \quad (5)$$

Coefficients  $b(t)$  and  $c(t)$  are to be determined by setting such conditions so as solution (5) observe equations (3) and (4):



**Fig.1.** Ingot positioning inside the pit furnace

$$b(t) = \frac{q_1(t)}{\lambda(T)} \quad (6)$$

$$c(t) = \frac{q_2(t) - q_1(t)}{2 \cdot L \cdot \lambda(T)} \quad (7)$$

From (5), (6) and (7) it results

$$T(y, t) = a_1(t) + \frac{q_1(t)}{\lambda(T)} \cdot y + \frac{q_2(t) - q_1(t)}{2 \cdot L \cdot \lambda(T)} \cdot y^2 \quad (8)$$

In order to facilitate the solving of the problem, we resorted to a simplifying hypothesis, which consists in considering the thermo-physical parameters as constant along certain time intervals whose span and number shall be chosen later in a convenient way. So, if we suppose  $\lambda = ct.$ ,  $c_p = ct.$ , and  $\rho = ct.$  along the intervals, we have:

$$T(y, t) = \frac{1}{\lambda} \cdot \left[ \lambda \cdot a_1(t) + q_1(t) \cdot y + \frac{q_2(t) - q_1(t)}{2 \cdot L} \cdot y^2 \right] \quad (9)$$

We note:  $\lambda \cdot a_1(t) = a(t)$  and by deriving the expression (9) with respect to time we obtain:

$$\frac{\partial T}{\partial t}(y, t) = \frac{1}{\lambda} \cdot \left[ a'(t) + q_1'(t) \cdot y + \frac{q_2'(t) - q_1'(t)}{2 \cdot L} \cdot y^2 \right] \quad (10)$$

We then derive the same expression (9) with respect to  $y$  and we have:

$$\lambda \cdot \frac{\partial T}{\partial y}(y, t) = q_1(t) + \frac{q_2(t) - q_1(t)}{L} \cdot y \quad (11)$$

$$\frac{\partial}{\partial y} \left[ \lambda \cdot \frac{\partial T}{\partial y}(y, t) \right] = \frac{q_2(t) - q_1(t)}{L} \quad (12)$$

In order to find the expression of  $a(t)$  we impose the checking condition for Galerkin integral:

$$\int_0^L \left\{ \rho(T) \cdot c_p(T) \cdot \frac{\partial T}{\partial t}(y, t) - \frac{\partial}{\partial y} \left[ \lambda(T) \cdot \frac{\partial T}{\partial y}(y, t) \right] \right\} \delta T dy = 0 \quad (13)$$

But as  $\lambda$ ,  $c_p$  and  $\rho$  are constant along the time intervals, considering (10) and (12) Galerkin integral becomes:

$$\delta T \cdot \int_0^L \left\{ \frac{\rho \cdot c_p}{\lambda} \cdot \left[ a'(t) + q_1'(t) \cdot y + \frac{q_2'(t) - q_1'(t)}{2 \cdot L} \cdot y^2 \right] - \frac{q_2(t) - q_1(t)}{L} \right\} dy = 0 \tag{14}$$

In order that expression (14) be true irrespective of the values taken by  $\delta T$  the Galerkin integral has to be null, which means:

$$\frac{\rho \cdot c_p}{\lambda} \cdot \left[ a'(t) \cdot L + q_1'(t) \cdot \frac{L^2}{2} + \frac{q_2'(t) - q_1'(t)}{2 \cdot L} \cdot \frac{L^3}{3} \right] - \frac{q_2(t) - q_1(t)}{L} \cdot L = 0 \tag{15}$$

and

$$a'(t) = \frac{\lambda \cdot [q_2(t) - q_1(t)]}{\rho \cdot c_p \cdot L} - q_1'(t) \cdot \frac{L}{2} - \frac{q_2'(t) - q_1'(t)}{6} \cdot L \tag{16}$$

The integral of relation (16) with respect to time is:

$$a(t) = \frac{\lambda}{\rho \cdot c_p \cdot L} \cdot \int [q_2(t) - q_1(t)] dt - \frac{L}{2} \cdot q_1(t) - \frac{L}{6} \cdot [q_2(t) - q_1(t)] + C \tag{17}$$

In order to determine the heat fluxes  $q_1(t)$  and  $q_2(t)$  we considered a variation of the temperature in the main pit furnace with respect to time,  $T_i(t)$ , as given in figure 2, and the variations  $T(0,t)$  respectively  $T(L,t)$  were considered linear variations.

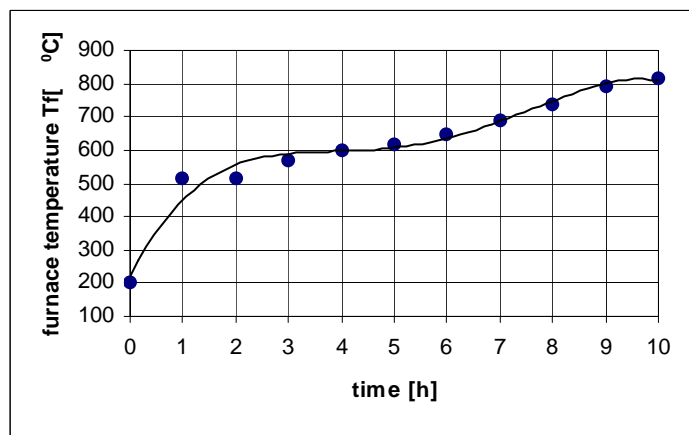


Fig.2. The time variation of temperature inside the pit furnace

If we mark:

$$\int q_1(t) dt = g_1(t) ; \int q_2(t) dt = g_2(t) \tag{18}$$

and we introduce (17) in (9) we obtain the final expression for the approximate temperature distribution in the ingot cross section (for a certain time interval):

$$T(y,t) = \frac{q_2(t) - q_1(t)}{\lambda} \cdot \left( \frac{y^2}{2 \cdot L} - \frac{L}{6} \right) + \frac{g_2(t) - g_1(t)}{\rho \cdot c_p \cdot L} + \frac{q_1(t)}{\lambda} \cdot \left( y - \frac{L}{2} \right) + \frac{C}{\lambda} \tag{19}$$

As we have already mentioned, in order that hypothesis:  $\lambda = ct.$ ,  $c_p = ct.$ , and  $\rho = ct.$  (along intervals) be as plausible as possible, we have to consider the time interval divided into several shorter intervals. These intervals have been chosen so as the modifications of the thermo-physical property values along each should be insignificant. As a result of such hypotheses, the integration constant of (19) shall be modified for each interval. Anyway, it is obvious that the temperature (of a point on the ingot cross-section) at the end of one time interval is identical to the temperature at the beginning of the next time interval:

$$T_{n+1}(y, n\Delta t) = T_n(y, n\Delta t) \tag{20}$$

where  $n = 0,1,2,\dots$  represents the number of the time interval, and  $\Delta t$  is the width of this time interval. Combining (19) and (20) we obtain the formula for the determination of the temperature function along the interval under consideration (n+1):

$$T_{n+1}(y,t) = T_n(y,n\Delta t) + \frac{1}{\lambda_{n+1}} [q_2(t) - q_2(n\Delta t) + q_1(n\Delta t) - q_1(t)] \left( \frac{y^2}{2L} - \frac{L}{6} \right) + \frac{1}{\rho_{n+1} \cdot c_{p_{n+1}} \cdot L} [g_2(t) - g_2(n\Delta t) + g_1(n\Delta t) - g_1(t)] + \frac{1}{\lambda_{n+1}} [q_1(t) - q_1(n\Delta t)] \left( y - \frac{L}{2} \right) \tag{21}$$

The total heating time is 10 hours. This interval has been divided into 10 equal parts, so that  $\Delta t = 1$  hour. The real variations of thermal conductivity  $\lambda$  and specific heat  $c_p$  are given in figures 3 respectively 4. The value of the specific mass is considered to be constant along the entire heating span and equal to  $\rho=7800\text{Kg/m}^3$ .

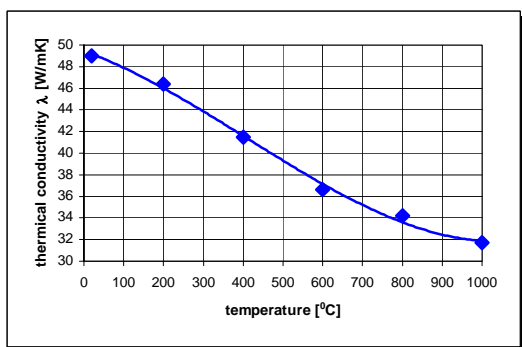


Fig.3. The real variation of thermal conductivity with respect to temperature

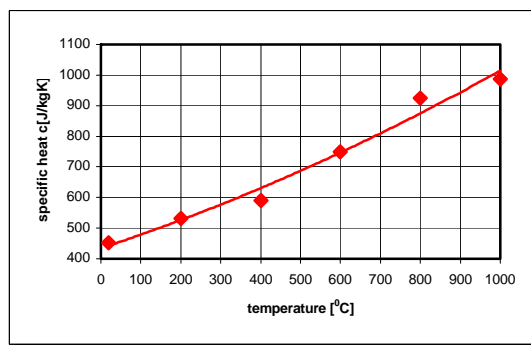


Fig.4. The real variation of specific heat with respect to temperature

### 3. RESULTS

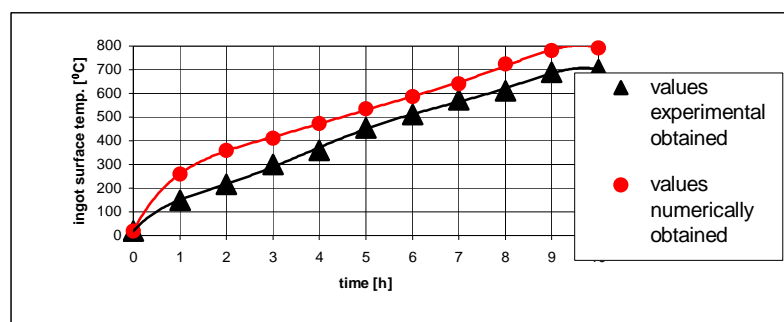
We hereinafter give the values we obtained by means of the method for the temperatures of points on the ingot surface ( $y = 0$ ) and inside it ( $y = L/2$ ). These values are compared to the temperature obtained by experimental measurements, at industrial scale.

### 4. CONCLUSIONS

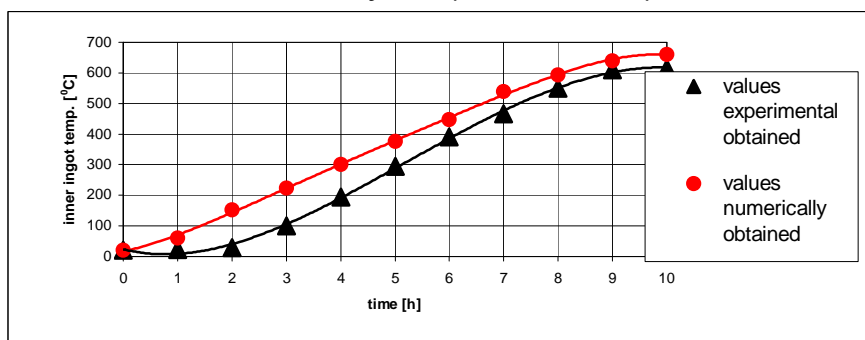
The analytically determined temperature values on the ingot surface overlap with the experimental ones by an error that differs according to the time interval of the heating up. This can be noticed in figure 5.

The reason these differences appear is determined by the way in which the heat transfer conditions have been determined, i.e. by convection and radiation, between the burned gases and the surface of the ingot. The choice of a global heat exchange coefficient  $\alpha$ , with a constant average value all along the heating time of the ingot surface, makes the model under consideration be closer to reality for the temperature range corresponding to a mean value of  $\alpha$ , i.e. in the medial zone of the total heating interval (400 – 600 °C), as can be noticed in figure 5.

The analytically obtained values of temperatures in the ingot core differ relatively much from the experimental ones, particularly in the first part of the heating time. This can be noticed in figure 6.



**Fig.5.** Temperatures of points belonging to the ingot cross-section for which  $y = 0$  (outer surface)



**Fig.6.** Temperatures of points belonging to the ingot cross-section for which  $y = L/2$  (ingot core)

The explanation of these errors resides in the fact that on establishing the thermo-physical properties of the material, the temperature variables (thermal conductivity  $\lambda$  and specific heat  $c$ ), we could only use some of their values, corresponding to the last heating time intervals. In order to cover the temperature interval corresponding to the entire heating time, we extrapolated the existent values and drew approximate variation curves for these magnitudes (see figures 3. and 4.). It is obvious that the precision of these curves is less than acceptable particularly in the area of low temperatures, and this is why the values calculated for temperatures in the core area of the ingot are subject to errors.

The explanation given above is based on the fact that inside the ingot, the heat transfer from hot areas to colder areas is done exclusively by thermal conduction, a phenomenon that is directly influenced by the thermo-physical properties of the material inside which they take place.

Finally, the errors can also be blamed, of course, on the way of constructing the mathematical model that describes the phenomenon, as well as on the method of solving it.

We can therefore conclude that the analytical model under consideration is capable of furnishing its user relatively exact results with respect to the evolution of the temperature field inside a field of analysis, provided the initial data be closest to the practical ones, and that the limit conditions, initial or spatial, be least simplified, so that the numeric model emulate as minutely as possible the real phenomenon.

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