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SECONDARY FLOW CAUSED BY A ROTATIONAL ELECTROMAGNETIC FORCE FIELD

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The development of secondary flows is a salient feature of a fluid flow in the presence of rotational force fields. Examples of rotational forces are the shear force in a viscous fluid, the electromagnetic force in an electrically conducting fluid, the pressure force in a stratified fluid, and the Coriolis force experienced by a flow in rotation. When a secondary flow occurs, the perturbated fluid motion has a magnitude of the same order as that of the primary flow. Therefore, the secondary flow is a nonlinear phenomenon, which is contained only in the solution of the governing equations whose nonlinear terms are retained.

When an electric current is passed through a conducting fluid, a magnetic field is generated. In the laboratory under a steady state, even if the fluid medium is in motion, the magnetic field H and the current density J are governed approximately by the following equations,

$$J = \nabla \times H$$

$$\nabla \times H \times H = 0$$

The governing equations cannot be independent of the fluid velocity V in the problems of astronomical scales. In the presence of a current and a magnetic field, the conducting fluid is subject to an electromagnetic body force per unit volume of the form

$$f = J \times \mu_m H$$

where μ_m is the magnetic permeability of the fluid medium. Unless the current is uniform or is in some simple geometry, such as a two-dimensional configuration, this body force is in general rotational; that is, $\nabla \times f \neq 0$. This rotational force causes many interesting phenomena in a conducting flow that are not commonly found in the ordinary fluid flows.

The equation of motion for an incompressible conducting fluid is obtained by simply adding the above electromagnetic force term, to the right-hand side of the Navier-Stokes equation. Rewriting the equation with

$$\nabla \times V = \zeta$$

being the vorticity, we have

$$\frac{\partial V}{\partial t} - V \times \zeta = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}V^2\right) - \nu\nabla \times \zeta + \frac{\mu_m}{\rho} J \times H$$

in which $\nu = \mu/\rho$ is the kinematic viscosity. Taking the curl of this vector equation and substituting J from its above definition yields

$$\frac{\partial\zeta}{\partial t} - \nabla \times (\nabla \times \zeta) = -\nu\nabla \times \nabla \times \zeta + \frac{\mu_m}{\rho}\nabla \times [(\nabla \times H) \times H]$$

This equation describes the transportation and product of vorticity in a conducting viscous incompressible fluid. The continuity equation for such a fluid remains the same form as that for nonconducting fluid.

$$\nabla \cdot V = 0$$

The procedure for solving a magnetohydrodynamic problem in the laboratory scale is outlined as follows. The steady electromagnetic fields are first determined from the previous definitions utilizing their appropriate boundary conditions specified by the problem. Upon substitution of H into the last term of Navier-Stokes equation, the effect of the electromagnetic force is computed as a known function of spatial coordinates. The solution V to this equation, satisfying the specified fluid boundary conditions, is then sought under the constraint imposed by the equation of continuity.

As a specific example let us compute the motion of a conducting fluid flowing through a circular tube of radius r_0 in the presence of a converging-diverging current flow. Such a current distribution is obtained by applying potential differences across a small screen electrode of radius r_1 and two large screen electrodes covering the ends of the nonconducting tube. The electrodes, through which the fluid can freely move, are separated by the same distance z_0. In the presence of a conducting fluid, electric current will flow along the tube to form a constriction at the central electrode. When it interacts with the associated magnetic field in the azimuthal direction, this current produces a rotational force field that will modify the originally uniform motion of the fluid. In addition, the electromagnetic force, $J \times \mu_m H$, has a component pointing toward the tube axis that causes a *"pinch effect"* to build up the pressure near the axis. An analysis of the magnetohydrostatic pinch due to the uniform current can be found in Section 6.13 of Hughes and Young [1]. In our problem the pinch is nonuniform; it has a strong effect in the neighborhood of the central electrode. Thus the fluid is moving through a pressure hump created electromagnetically. The situation is somewhat analogous to the situation in which plasma is blowing toward an electric arc.

Because of the axisymmetric geometry of the flow, the stream function can again be used, so that the continuity equation is satisfied and the axial and radial velocity components are expressed as

$$u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}$$

The magnetic field has only one component, h_θ, in the azimuthally direction if the current flow is axisymmetric.

Let $\pi r_0^2 u_0$ and $\pi r_0^2 J_0$ to be the volume flow rate and the total current, respectively, passing through the tube. By using r_0, u_0 and J_0 as the reference quantities, the following dimensionless variables can be introduced.

$$R = \frac{r}{r_0}, \quad Z = \frac{z}{r_0}, \quad U = \frac{u_z}{u_0}, \quad V = \frac{u_r}{u_0}$$

$$T = \frac{t}{r_0/u_0}, \quad \Psi = \frac{\psi}{r_0^2 u_0}, \quad \Omega = \frac{-\zeta}{u_0/r_0}, \quad H = \frac{h_\theta}{r_0 J_0}$$

In terms of these new variables, the governing equations for an axisymmetric configuration become

$$\begin{aligned}
&(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2})RH = 0 \\
&U = \frac{1}{R}\frac{\partial \Psi}{\partial R} \\
&V = -\frac{1}{R}\frac{\partial \Psi}{\partial Z} \\
&(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2})\Psi = R\Omega
\end{aligned} \qquad (*)$$

$$\begin{aligned}
\frac{\partial \Omega}{\partial T} = &-\frac{\partial (U\Omega)}{\partial Z} - \frac{\partial (V\Omega)}{\partial R} + C\frac{H}{R}\frac{\partial H}{\partial Z} \\
&+ \frac{1}{Re}(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} - \frac{1}{R^2} + \frac{\partial^2}{\partial Z^2})\Omega
\end{aligned} \qquad (**)$$

The two dimensionless parameters in the last equation are $C(\equiv \mu_m r_0^2 J_0^2 \Big/ \frac{1}{2}\rho u_0^2)$, the magnetic pressure number, and $Re(\equiv u_0 r_0/\nu)$, the Reynolds number of the flow.

Simple boundary conditions are chosen for our analysis. For example, instead of spending time on computing the growth of the boundary layer along the tube wall, which is a phenomenon not essential to the present problem, a slip boundary condition is assumed there that states that the wall coincides with the outermost streamline and that along this particular streamline vorticity vanishes. The boundary conditions for the electromagnetic field are that the current must be tangent to the nonconducting tube wall and the radial component of the current flow vanishes at both the end and central electrodes.

Since the electromagnetic field is symmetric about the central electrode at $z = 0$, we need only to find the solution in the region $-z_0 \le z \le 0$ to the left of the central electrode. The mirror image of this solution gives that in the region to the right. The boundary conditions in terms of RH are listed below.

$$R = 0 \quad \text{and} \quad -\frac{z_0}{r_0} \le Z \le 0: \quad RH = 0$$

$$R = 1 \quad \text{and} \quad -\frac{z_0}{r_0} \le Z \le 0: \quad RH = \frac{1}{2}$$

$$0 < R < \frac{r_1}{r_0} \quad \text{and} \quad Z = 0: \quad \frac{\partial RH}{\partial Z} = 0$$

$$\frac{r_1}{r_0} \le R < 1 \quad \text{and} \quad Z = 0: \quad RH = \frac{1}{2}$$

$$\frac{r_1}{r_0} \le R < 1 \quad \text{and} \quad Z = 0: \quad RH = \frac{1}{2}$$

When there is a net flow through the tube, the flow pattern, unlike the electromagnetic field, is not symmetric about the central electrode, so the whole region between the two electrodes must be considered. By assuming that the flow is purely axial when passing through the end screen electrodes, the fluid boundary conditions are as follows,

$$R = 0 \quad -\frac{z_0}{r_0} \le Z \le \frac{z_0}{r_0}: \quad \Psi = \Omega = 0$$

$$R = 1 \quad -\frac{z_0}{r_0} \le Z \le \frac{z_0}{r_0}: \quad \Psi = \frac{1}{2}, \Omega = 0$$

$$0 < R < 1 \quad \text{and} \quad Z = \pm\frac{z_0}{r_0}: \quad \frac{\partial \Psi}{\partial Z} = \frac{\partial \Omega}{\partial Z} = 0$$

The system of equations, including the nonlinear equation (**), is to be solved numerically for flows whose Reynolds numbers are not small. These problems generally cannot be solved analytically by using linearization techniques. The solution will be obtained at a finite number of grid points having coordinates $Z_i = (i-2)\Delta Z - z_0/r_0$, $R_j = (j-1)\Delta R$ and at discrete time steps separated by constant intervals ΔT. In the present work ΔZ and ΔR are chosen to have the same value h; this is not, however, generally necessary.

To replace the governing differential equations by finite-difference equations, evaluated at (Z_i, R_i), we approximate all linear spatial derivatives by using the central-difference formulas. Thus, the first four equations (*) become

$$(RH)_{i,j} = \frac{1}{4}[(RH)_{i+1,j} + (RH)_{i-1,j} + \alpha_j (RH)_{i,j+1} + \beta_j (RH)_{i,j-1}]$$

$$U_{i,j} = \frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{2hR_j}$$

$$V_{i,j} = \frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{2hR_j}$$

$$\Psi'_{i,j} = \frac{1}{4}(\Psi'_{i+1,j} + \Psi'_{i-1,j} + \alpha_j \Psi'_{i,j+1} + \beta_j \Psi'_{i,j-1} + h^2 R_j \Omega'_{i,j})$$

where

$$\alpha_j = 1 - \frac{h}{2R_j} = \frac{j-1.5}{j-1}$$

$$\beta_j = 1 + \frac{h}{2R_j} = \frac{j-0.5}{j-1}$$

The (i,j) subscripts denote quantities evaluated at the grid point (Z_i, R_j), while the notation $(RH)_{i,j}$ is specifically used to replace $R_j H_{i,j}$. The unprimed and primed flow variables represented, respectively, those at two consecutive time levels T and $T + \Delta T$.

Along the tube axis the velocity is calculated according to the formula

$$U_{i,1} = \frac{2\Psi_{i,2}}{h^2}$$

This relation is derived from (*) by assuming constant velocity within a small radial distance h from the axis.

Care must be taken in approximating the equation of motion (**). The first two nonlinear terms on the right-hand side are similar to the substantial derivative; they will be replaced by the upwind-difference scheme considered in the previous section. The scheme must be modified, however, to accommodate the arbitrary local flow direction, which may vary from one grid point to another. We adopt here a scheme developed by Torrance (1968) that has been successfully used in solving natural convection (Torrance and Rockett, 1969) and rotating flow (Kopecky and Torrance, 1973) problems.

We first define U_f and U_b as the average axial velocities evaluated, respectively, at half a grid forward and backward from the point (Z_i, R_j) in the Z direction, or,

$$U_f = \frac{1}{2}(U_{i+1,j} + U_{i,j})$$

$$U_b = \frac{1}{2}(U_{i,j} + U_{i-1,j})$$

Similarly, we define

$$V_f = \frac{1}{2}(V_{i,j+1} + V_{i,j})$$

$$V_b = \frac{1}{2}(V_{i,j} + V_{i,j-1})$$

as the radial velocities averaged at the half a grid forward and backward from that point in the R direction, repectively. It can be easily be verified that the upwind differencing form is automatically preserved when the following numerical formulas are used.

$$\left[\frac{\partial(U\Omega)}{\partial Z}\right]_{i,j} = \frac{1}{2h}[(U_f - |U_f|)\Omega_{i+1,j} + (U_f + |U_f| - U_b + |U_b|)\Omega_{i,j} - (U_b + |U_b|)\Omega_{i-1,j}]$$

$$\left[\frac{\partial(V\Omega)}{\partial R}\right]_{i,j} = \frac{1}{2h}[(V_f - |V_f|)\Omega_{i,j+1} + (V_f + |V_f| - V_b + |V_b|)\Omega_{i,j} - (V_b + |V_b|)\Omega_{i,j-1}]$$

The remaining terms in (**) are approximated by using forward differencing in time and central differencing in space. Their individual expressions follow.

$$\left(\frac{\partial\Omega}{\partial T}\right)_{i,j} = \frac{\Omega'_{i,j} - \Omega_{i,j}}{\Delta T}$$

$$\left(\frac{H}{R}\frac{\partial H}{\partial Z}\right)_{i,j} = \left(\frac{RH}{R^3}\frac{\partial RH}{\partial Z}\right)_{i,j} = (RH)_{i,j}\frac{(RH)_{i+1,j} - (RH)_{i-1,j}}{2hR_j^3}$$

$$\left(\frac{\partial^2\Omega}{\partial R^2}+\frac{1}{R}\frac{\partial\Omega}{\partial R}-\frac{\Omega}{R^2}+\frac{\partial^2\Omega}{\partial Z^2}\right)_{i,j}=\frac{1}{2hR_j}(\Omega_{i,j+1}-\Omega_{i,j-1})$$
$$+\frac{1}{h^2}\left[\Omega_{i,j+1}+\Omega_{i,j-1}+\Omega_{i+1,j}+\Omega_{i-1,j}-\left(4+\frac{h^2}{R_j^2}\right)\Omega_{i,j+1}\right]$$

Finally it results an equation that enables us to compute Ω at $T+\Delta T$ based on the dimensionless vorticity distribution Ω at the previous time step T.

In this way the fluid equations governing an unsteady flow have been formulated for numerical solution. To find the steady-state solution for the tube flow passing through a current constriction, a uniform axial fluid flow is first assumed. The finite-difference equations are then used repeatedly to compute the flow conditions at increasing time steps, until the difference between the solutions at two consecutive steps becomes negligibly small. When this happens the result is considered to be steady-state solution for which we are looking. Thus we replace the original problem by an unsteady flow problem concerning the development of a uniform flow after a converging-diverging current field is suddenly turned on. The marching procedure described here is a powerful technique that is quite commonly employed in fluid dynamics to find the steady-state solution.

However, numerical instability may occur in the process of approaching the steady-state solution if the size of the time increment is not properly chosen. For a given grid size h, the constraints on ΔT are to be derived from a quasilinear analysis of Lax and Richtmyer (1956) for linear equations instead of from Hirt's stability analysis. The latter is not adequate for the present case. As a first step, the difference representation of (**) is written in the form

$$\Omega'_{i,j}=a_1\Omega_{i+1,j}+a_2\Omega_{i-1,j}+a_3\Omega_{i,j}+a_4\Omega_{i,j+1}+a_5\Omega_{i,j-1}+b$$

in which the coefficients a_k , are expressed in terms of velocity components at time T, are considered constant over the period ΔT in the computation, and b is a constant determined by the magnetic field. According to Lax and Richtmyer, the numerical scheme is stable if the coefficients a_k are all positive. It can be shown that this requirement is automatically satisfied for all the coefficients except a_3. To require that a_3 be greater than or equal to zero, a condition is derived that specifies the upper limit for the time increment.

$$\Delta T\le\left[\frac{1}{2h}(U_f+|U_f|-U_b+|U_b|+V_f+|V_f|-V_b+|V_b|)+\frac{1}{\mathrm{Re}\,h^2}\left(4+\frac{h^2}{R_j^2}\right)\right]^{-1}$$

Although the current density does not appear explicitly in this relation, its effect is shown indirectly through the velocities U_f, U_b, V_f and V_b defined above. In general, a higher current density causes higher velocities in the neighborhood of the central electrode and, therefore, needs a smaller ΔT in the computation.

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