ANNALS OF THE FACULTY OF ENGINEERING HUNEDOARA
2005, Tome III, Fascicole 2

AN EXTENSION OF YOSIDA APPROXIMATION

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ABSTRACT:
The aim of this note is to formulate an extension of Yosida approximation for infinitesimal generator of a strongly continuous semigroup. Some properties and applications of the extended Yosida approximation will also be discussed.
A.M.S.(2000) Subject Classification: 47D03, 47D06, 47D62.

KEYWORDS:
C_0-semigroup, the extended Yosida approximation, once-integrated semigroup

1. PRELIMINARY

Let E be a complex Banach space. We denote by $B(E)$ the Banach algebra of bounded linear operators on E. For a closed linear operator A, not necessary bounded, with domain D(A) in the space E, denote by ρ(A) and R(.;A) the resolvent set of A and the resolvent of A, respectively.

Definition 1.1 A C_0-semigroup on a Banach space E is a family $\{T(t)\}_{t\geq 0} \subset B(E)$ of bounded linear operators such that:

i) T(0) = I;
ii) T(t+s) = T(t)T(s), $\forall$ t,s > 0;
iii) $\lim_{t\to 0} T(t)x = x$, $\forall x \in E$

The most important object associated to a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is its generator.

Definition 1.2 By the generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ we understand a linear operator A (possibly unbounded) defined on the domain

$$D(A) = \left\{ x \in E \,\middle|\, \lim_{t\to 0} \frac{T(t)x - x}{t} \ \text{ exists} \right\} \tag{1}$$

by:

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \forall x \in D(A). \tag{2}$$

It is not difficult to show that for every C_0-semigroup $\{T(t)\}_{t\geq 0}$ there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq M \cdot e^{\omega \cdot t}, \ \forall t \geq 0 \tag{3}$$

So we denote by $SG(M,\omega)$ the set of this C_0-semigroups which is said to be exponentially bounded.

For more information about C_0-semigroups we refer to [Pa'83], [Vr'01], [Yo'67] and the references therein.

2. THE EXTENDED YOSIDA APPROXIMATION OF INFINITESIMAL GENERATOR

For $\omega \geq 0$ we denote by A_ω the set

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\}. \tag{4}$$

The next Lemma follows from [Pa'83, Lemma 3.2, p.9].

Lemma 2.1 Let $A : D(A) \subset E \to E$ be a linear (unbounded) operator which satisfy the conditions:

i) A is closed and $\overline{D(A)} = E$;

ii) there exists constants $\omega \geq 0$ and $M \geq 1$ such that $A_\omega \subset \rho(A)$ and for every $\lambda \in A_\omega$, we have:

$$\left\| R(\lambda; A)^n \right\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad \forall n \in N^*. \tag{5}$$

Then for all $\lambda \in A_\omega$, we have:

$$\lim_{\operatorname{Re}\lambda \to \infty} \lambda R(\lambda; A)x = x, \quad \forall x \in E. \tag{6}$$

Moreover, $\lambda A R(\lambda; A) \in B(E)$ and:

$$\lim_{\operatorname{Re}\lambda \to \infty} \lambda A R(\lambda; A)x = Ax, \quad \forall x \in D(A). \tag{7}$$

Proof. Let $x \in D(A)$ and $\lambda \in C$ be such that $\operatorname{Re}\lambda > \omega$. Then $R(\lambda; A)(\lambda I - A)x = x$. For $\operatorname{Re}\lambda \to \infty$, we have:

$$\begin{aligned} \|\lambda R(\lambda; A)x - x\| &= \| R(\lambda; A)x \| \leq \| R(\lambda; A) \| \cdot \| Ax \| \leq \\ &\leq \frac{M}{\operatorname{Re}\lambda - \omega} \| Ax \| \to 0 \end{aligned} \tag{8}$$

from where one deduce that:

$$\lim_{\operatorname{Re}\lambda \to \infty} \lambda R(\lambda; A)x = x, \quad \forall x \in D(A). \tag{9}$$

For $x \in E$, because $\overline{D(A)} = E$, there exists a sequence $(x_n)_{n \in N} \subset D(A)$ such that $x_n \to x$ and $n \to \infty$. We obtain:

$$\begin{aligned} &\|\lambda R(\lambda; A)x - x\| \leq \|\lambda R(\lambda; A)x - \lambda R(\lambda; A)x_n\| + \|\lambda R(\lambda; A)x_n - x_n\| + \\ &+ \|x_n - x\| \leq \frac{|\lambda| M + \operatorname{Re}\lambda - \omega}{\operatorname{Re}\lambda - \omega} \|x_n - x\| + \frac{M}{\operatorname{Re}\lambda - \omega} \|Ax_n\| \end{aligned} . \tag{10}$$

But $x_n \to x$ as $n \to \infty$. Hence for any $\varepsilon > 0$, there exists $n_\varepsilon \in N$ such that:

$$\|x_n - x\| < \varepsilon \frac{\operatorname{Re}\lambda - \omega}{|\lambda| M + \operatorname{Re}\lambda - \omega}. \tag{11}$$

Consequently:

$$\left\|\lambda R(\lambda;A)x - x\right\| < \varepsilon + \frac{M}{\mathrm{Re}\lambda - \omega}\left\|Ax_{n_\varepsilon}\right\| \tag{12}$$

from where it follows that

$$\lim_{\mathrm{Re}\lambda\to\infty}\left\|\lambda R(\lambda;A)x - x\right\| < \varepsilon, \quad \forall x \in E, \tag{13}$$

that is:

$$\lim_{\mathrm{Re}\lambda\to\infty}\lambda R(\lambda;A)x = x, \quad \forall x \in E. \tag{14}$$

Moreover:

$$\lambda AR(\lambda;A) = \lambda[\lambda I - (\lambda I - A)]R(\lambda;A) = . \\ = \lambda[\lambda R(\lambda;A) - I] = \lambda^2 R(\lambda;A) - \lambda I \tag{15}$$

Than we have:

$$\left\|\lambda AR(\lambda;A)x\right\| = \left\|\lambda[\lambda R(\lambda;A) - I]x\right\| \le |\lambda|\left\|\lambda R(\lambda;A)x - x\right\| \le \\ \le |\lambda|\left(\left\|\lambda R(\lambda;A)x\right\| + \left\|x\right\|\right) \le |\lambda|\left(\frac{|\lambda|M}{\mathrm{Re}\lambda - \omega} + 1\right)\left\|x\right\|, \qquad \forall x \in E \tag{16}$$

so we deduce that $\lambda AR(\lambda;A) \in B(E)$.

If $x \in D(A)$, then it fallows:

$$\lambda R(\lambda;A)Ax = \left[\lambda^2 R(\lambda;A) - \lambda I\right]x = \lambda AR(\lambda;A)x, \tag{17}$$

from where we conclude that:

$$\lim_{\mathrm{Re}\lambda\to\infty}\lambda AR(\lambda;A)x = \lim_{\mathrm{Re}\lambda\to\infty}\lambda R(\lambda;A)Ax = Ax, \qquad \forall x \in D(A). \square \tag{18}$$

Remark 2.2 We can say that the bounded operators $\lambda AR(\lambda;A)$ are some approximations of operator A which is unbounded.

So we can introduce the next definition, which generalize [Pa'83, p.9].

Definition 2.3 The family $\{A_\lambda\}_{\lambda\in A_\omega} \subset B(E)$, where $A_\lambda = \lambda AR(\lambda;A)$ is called the extended Yosida approximation of A.

Following the same method in proving [Pa'83, Lemma 7.2, p.26], we can prove

Theorem 2.4 Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0} \in SG(M,\omega)$ and let $\{A_\mu\}_{\mu\in A_\omega}$ be the extended Yosida approximation of A. Then for all $\mu \in A_\omega$ there is a constant $\Omega > \omega$ such that $A_\Omega \subset \rho(A_\mu)$ and for each $\lambda \in A_\Omega$ we have:

$$\left\|R(\lambda;A_\mu)\right\| \le \frac{M}{\mathrm{Re}\lambda - \Omega}. \tag{19}$$

Moreover, for $\varepsilon > 0$, there is a constants C>0 (depending only on M and ε) such that for every $x \in D(A)$,

$$\left\|R(\lambda;A_\mu)x\right\| \le \frac{C}{|\lambda|}\left(\left\|x\right\| + \left\|Ax\right\|\right), \tag{20}$$

for all $\lambda,\mu \in C$, with $\mathrm{Re}\lambda > \Omega + \varepsilon$ and $\mathrm{Re}\mu > \omega + \frac{|\mu|}{2}$.

Proof. Let $\mu \in A_\omega$ be fixed. We note that A_μ is the infinitesimal generator of the uniformly continuous semigroup $\left\{e^{tA_\mu}\right\}_{t\geq 0}$.
We obtain:

$$\left\| e^{tA_\mu} \right\| = \left\| e^{t(\mu^2 R(\mu;A)-\mu I)} \right\| = \left\| e^{-\mu t I} e^{\mu^2 t R(\mu;A)} \right\| \leq e^{-\operatorname{Re}\mu t} \left\| \sum_{k=0}^{\infty} \frac{t^k \mu^{2k} R(\mu;A)^k}{k!} \right\| \leq$$

$$\leq e^{-\operatorname{Re}\mu t} \sum_{k=0}^{\infty} \frac{t^k |\mu|^{2k} \left\| R(\mu;A)^k \right\|}{k!} \leq e^{-\operatorname{Re}\mu t} \sum_{k=0}^{\infty} \frac{t^k |\mu|^{2k} M}{k!(\operatorname{Re}\mu - \omega)^k} =$$

$$= M e^{-\operatorname{Re}\mu t} \sum_{k=0}^{\infty} \frac{\left(\frac{t|\mu|^2}{\operatorname{Re}\mu - \omega} \right)^k}{k!} = M e^{-\operatorname{Re}\mu t} e^{\frac{t|\mu|^2}{\operatorname{Re}\mu - \omega}} = M e^{\frac{\omega \operatorname{Re}\mu + \operatorname{Im}^2 \mu}{\operatorname{Re}\mu - \omega} t}. \tag{21}$$

If we denote:

$$\Omega = \frac{\omega \operatorname{Re}\mu + \operatorname{Im}^2 \mu}{\operatorname{Re}\mu - \omega} \tag{22}$$

then it fallows that:

$$\Omega = \omega + \frac{\omega^2 + \operatorname{Im}^2 \mu}{\operatorname{Re}\mu - \omega} > \omega \tag{23}$$

and that $A_\Omega = \left\{ \lambda \in C \,\middle|\, \operatorname{Re}\lambda > \Omega \right\} \subset \rho(A\mu)$.
Furthermore, for each $\lambda \in A_\Omega$ we obtain:

$$\left\| R(\lambda; A_\mu) \right\| \leq \frac{M}{\operatorname{Re}\lambda - \Omega}. \tag{24}$$

For $\lambda \in C$ with property $\operatorname{Re}\lambda > \Omega + \varepsilon, \varepsilon > 0$, it fallows that:

$$\left\| R(\lambda; A_\mu) \right\| \leq \frac{M}{\varepsilon}. \tag{25}$$

If $x \in D(A)$ and $\mu \in A_\omega$ is such that $\operatorname{Re}\mu > \omega + \frac{|\mu|}{2}$, than:

$$\left\| A_\mu x \right\| = \left\| \mu R(\mu;A) A x \right\| \leq |\mu| \left\| R(\mu;A) \right\| \left\| Ax \right\| \leq .$$

$$\leq |\mu| \frac{M}{\operatorname{Re}\mu - \omega} \left\| Ax \right\| \leq 2M \left\| Ax \right\| \tag{26}$$

By equality:

$$(\lambda I - A_\mu) R(\lambda; A_\mu) = I, \tag{26}$$

we deduce that:

$$R(\lambda; A_\mu) = \frac{1}{\lambda} I + \frac{1}{\lambda} R(\lambda; A_\mu) A_\mu \tag{27}$$

and therefore:

$$\left\| R(\lambda; A_\mu) x \right\| \leq \frac{1}{|\lambda|} \left(\|x\| + \left\| R(\lambda; A_\mu) \right\| \left\| A_\mu x \right\| \right) \leq$$

$$\leq \frac{1}{|\lambda|} \left(\|x\| + \frac{2M^2}{\varepsilon} \|Ax\| \right) \leq \frac{C}{|\lambda|} \left(\|x\| + \|Ax\| \right), \quad \forall x \in D(A) \tag{28}$$

where the constants C depends only on M and ε.□
The next theorem is a generalization of [DM'88, p.312].

Theorem 2.5 Let A be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0} \in SG(M,\omega)$, let $\{A_\mu\}_{\mu\in A_\omega}$ be the extended Yosida approximation of A and $\lambda \in C$ such that $\mathrm{Re}\lambda > \omega+\varepsilon$ is fixed for $\varepsilon > 0$. Then there exists $\mu \in A_\omega$ such that $\lambda \in \rho(A_\mu), \frac{\lambda\mu}{\lambda+\mu} \in \rho(A)$ and

$$R(\lambda; A_\mu) = \frac{1}{\lambda+\mu} I + \left(\frac{\mu}{\lambda+\mu}\right)^2 R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right). \tag{30}$$

Proof. By the Theorem 2.4, we deduce that for $\nu \in A_\omega$, there is $\Omega > \omega$ such that $A_\Omega \subset \rho(A_\nu)$. We have:

$$\Omega = \frac{\omega \mathrm{Re}\nu + \mathrm{Im}^2 \nu}{\mathrm{Re}\nu - \omega}. \tag{31}$$

Thus the inequality $\mathrm{Re}\lambda > \Omega$ is equivalent to:

$$\mathrm{Re}\lambda > \omega + \frac{\omega^2 + \mathrm{Im}\nu}{\mathrm{Re}\nu - \omega}. \tag{32}$$

Let $\varepsilon > 0$. If $\mu \in A_\omega$ is such that $\frac{\omega^2 + \mathrm{Im}\mu}{\mathrm{Re}\nu - \omega} < \varepsilon$, then $\mathrm{Re}\lambda > \omega + \varepsilon$ implies $\mathrm{Re}\lambda > \Omega$. Hence $\lambda \in \rho(A_\mu)$.

By the Theorem 2.4 it fallows:

$$\left\| R(\lambda; A_\mu) \right\| \leq \frac{M}{\mathrm{Re}\lambda - \omega}. \tag{33}$$

On the other hand, we have:

$$\begin{aligned} \mathrm{Re}\frac{\lambda\mu}{\lambda+\mu} &= \mathrm{Re}\left(\lambda - \frac{\lambda^2}{\lambda+\mu}\right) = \\ &= \mathrm{Re}\lambda - \mathrm{Re}\frac{\lambda^2}{\lambda+\mu} > \omega + \varepsilon - \mathrm{Re}\frac{\lambda^2}{\lambda+\mu} \end{aligned}. \tag{34}$$

Given k>0 such that $|\mathrm{Im}\lambda| \leq k$, we can find $\mu \in A_\omega$ such that $\mathrm{Re}\frac{\lambda^2}{\lambda+\mu} < \frac{\varepsilon}{2}$.

Then $\mathrm{Re}\frac{\lambda\mu}{\lambda+\mu} > \omega + \frac{\varepsilon}{2}$ and consequently $\frac{\lambda\mu}{\lambda+\mu} \in \rho(A)$.

We have:

$$\begin{aligned} &\frac{1}{\lambda+\mu}(\lambda I - A_\mu)(\mu I - A) R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right) = \\ &= \frac{1}{\lambda+\mu}[\lambda I - \mu^2 R(\mu; A) + \mu I](\mu I - A) R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right) = \\ &= \left[\mu I - A - \frac{\mu^2}{\lambda+\mu} I\right] R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right) = \left(\frac{\lambda\mu}{\lambda+\mu} I - A\right) R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right) = I \end{aligned} \tag{35}$$

Similarly

$$\frac{1}{\lambda+\mu}(\lambda I - A) R\left(\frac{\lambda\mu}{\lambda+\mu}; A\right)(\lambda I - A_\mu) = I \tag{36}$$

Therefore:

$$R(\lambda;A_\mu)=\frac{1}{\lambda+\mu}(\mu I-A)R\left(\frac{\lambda\mu}{\lambda+\mu};A\right). \tag{37}$$

Furthermore, using the resolving identity, it follows:

$$R\left(\frac{\lambda\mu}{\lambda+\mu};A\right)-R(\mu;A)=\left(\mu-\frac{\lambda\mu}{\lambda+\mu}\right)R\left(\frac{\lambda\mu}{\lambda+\mu};A\right)R(\mu;A). \tag{38}$$

from where we deduce that

$$\frac{1}{\lambda+\mu}(\mu I-A)R\left(\frac{\lambda\mu}{\lambda+\mu};A\right)=\frac{1}{\lambda+\mu}I+\left(\frac{\mu}{\lambda+\mu}\right)^2R\left(\frac{\lambda\mu}{\lambda+\mu};A\right).\square \tag{39}$$

3. APPLICATIONS OF THE EXTENDED YOSIDA APPROXIMATION

As an first application, using the extended Yosida approximation, we can prove a general version of the Hille-Yosida theorem.

Theorem 3.1 (Hille-Yosida) A linear operator: $A:D(A)\subset E\to E$ is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}\in SG(M,\omega)$ if and only if the following holds:

i) A is closed and $\overline{D(A)}=E$;

ii) there exists constants $\omega\geq 0$ and $M\geq 1$ such that $A_\omega\subset\rho(A)$ and for every $\lambda\in A_\omega$, we have:

$$\left\|R(\lambda;A)^n\right\|\leq\frac{M}{(\mathrm{Re}\lambda-\omega)^n},\quad\forall n\in N^*. \tag{40}$$

The basic idea in the proof (see [Le'03]) involves construction of a family of bounded linear operators (the extended Yosida approximation), approximating the unbounded operator A, which in the limit converge to the given operator A. The family of semigroups corresponding to the extended Yosida approximation are then shown to converge in the strong operator topology to a strongly continuous semigroup whose infinitesimal generator is the given operator A. Remark that we don't need to use here the standard procedure to renorming the Banach space E like in [Pa'83, Theorem 5.2 and Theorem 5,3].

The following result is an immediate consequence of Hille-Yosida theorem:

Corollary 3.2 Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}\in SG(M,\omega)$ and let $\{A_\lambda\}_{\lambda\in A_\omega}$ be the Yosida approximation of A. Then for each $x\in E$

$$T(t)x=\lim_{\mathrm{Re}\lambda\to\infty}e^{tA_\lambda}x, \tag{41}$$

uniformly for t in bounded intervals of $[0,\infty)$.

In the second application, we give an easy proof of a part of theorem due to Arendt [Ar'87, Theorem 4.1] on the integrated semigroups. This theory of integrated semigroups extends the powerful theory of strongly continuous semigroups to operators which do not satisfy the Hille-Yosida conditions. Integrated semigroups have been introduced in recent years and we refer to [Ar'87], [Hi'91], [KH'89], [MPV97], [Ne'88], [Th'90], and many others.

Definition 3.3 By non-degenerate once-integrated semigroup on E we understand a family $\{S(t)\}_{t\geq 0} \subset B(E)$ satisfying the following:

i) S(0)=0;

ii) the map $[0,\infty) \ni t \to S(t) \in B(E)$ is strongly continuous;

iii) for each t, s > 0 we have

$$S(t)S(s) = \int_0^t [S(\tau+s) - S(\tau)]d\tau . \quad (42)$$

iv) S(t)x = 0 for all $t \geq 0$ implies x = 0.

A non-degenerate once-integrated semigroup $\{S(t)\}_{t\geq 0} \subset B(E)$ is called exponentially bounded if there exists constants $M \geq 0 \text{ and } \omega \in R$ such that

$$\| S(t) \| \leq Me^{\omega t}, \quad \forall t \geq 0 \quad (43)$$

Definition 3.4 The generator $A : D(A) \subset E \to E$ of a non-degenerate once-integrated semigroup $\{S(t)\}_{t\geq 0} \subset B(E)$ is defined as follows: $x \in D(A)$ and Ax=y if and only if

$$S(t)x - tx = \int_0^t S(\tau)y\, d\tau, \quad \forall t \geq 0 .$$

(44)

We can give an elementary proof of a part of Arendt theorem about integrated semigroups using the extended Yosida approximation, the part of the generator (see [Pa'83, Definition 10.3, p.39]) and an idea of Adam Bobrowski [Bo'94, p.299].

Theorem 3.5(Arendt) Let

$$A : D(A) \subset E \to E \quad (45)$$

be a linear operator and $\omega \in R$. Suppose that:

i) A is closed;

ii) there exist $a \geq \max\{0,\omega\} \text{ and } M \geq 0 \text{ such that } A_a \subset \rho(A)$ and for each $\lambda \in A_a$ we have

$$\| R(\lambda; A)^n \| \leq \frac{M}{(\text{Re}\lambda - \omega)^n}, \quad \forall n \in N^* \quad . \quad (46)$$

Then A is the generator of a non-degenerate once-integrated semigroup $\{S(t)\}_{t\geq 0} \subset B(E)$ such that

$$\| S(t+h) - S(t) \| \leq Me^{\omega(t+h)} h, \quad \forall t,h \geq 0 . \quad (47)$$

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