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DIFFERENTIAL EQUATIONS IN SINGLE SPECIES MODELS

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ABSTRACT:
This study presents an introduction to dynamical population study. Here are exposed theoretical considerations upon single species models using differential equations and a classification of them. An example is considered and it is in totally solved using both analytical method and a Mathcad computing method.

KEYWORDS:
ecology, dynamical population, species, differential equation

1. INTRODUCTION

Ecology studies dynamical population, discussing various mathematical assumptions on the growth rate. These are intended to capture mathematically, in the simplest and efficient way, the dependence of the growth rate on food supply, spread disease, pollution, health condition and negative effects of overcrowding. If we hope to simulate this dynamics we need mathematical models which incorporate these interactions and these are requiring differential equations.

2. ONE SPECIES MODELS

The simplest of these models is a single species model, measuring the growth of one population [1,2]. We call P(t) the number of members in population $\mathcal{P}$ measured at the time t. The number P(t) is variously depending on the ecological laws influencing the growth of the population $\mathcal{P}$. We can obtain a diversity of differential equations for P(t) which depends on the initial conditions of the ecological laws.

The parameters in our mathematical model are:

- P(t) represents the number of members in population $\mathcal{P}$, where P(t) is a real-valued function of a real variable. We assume that P(t) has a continuous derivative.
- P'(t) represents the growth speed of the population $\mathcal{P}$ at the moment of time t.
- Thus the growth speed is:

$$P'(t) = \lim_{h \to 0} \frac{1}{h}\left[P(t+h) - P(t)\right] \tag{1}$$

$\frac{P'(t)}{P(t)}$ represents the growth rate of the population $\mathcal{P}$ at time t.

There are certain constants which are included in the model, depending on environment.

E1 Model: The population $\mathcal{P}$ growths such that the growth rate is constant for each time t. Let be $\frac{P'(t)}{P(t)} = a$, $a \in R$, a is fixed, with the initial condition $P(t_0) = P_0$. From our mathematical point of view we have a Cauchy problem:

$$P'(t) = aP(t) \quad , P(t_0) = P_0 \tag{2}$$

This is the case if the number of births and deaths in a small time period have a fixed ratio to the total population. These ratios will be linear functions, but independent of the size of the population (Figure 1). Integrating we obtain the formula for unlimited growth:

$$P(t) = e^{a(t-t_0)}P_0 \tag{3}$$

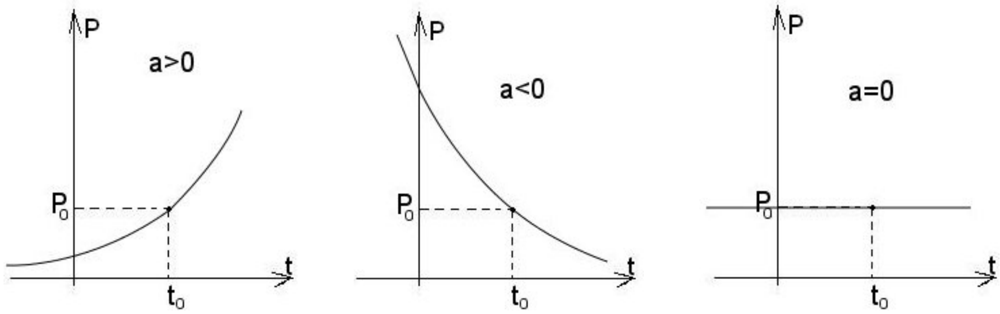

FIGURE 1. Unlimited growth model with fixed ratio

The growth rate can depend on many things.

E2 Model: Let us assume that σ is per capita existing food supply. There will be σ_0 a minimum necessary food supply to sustain the population. For $\sigma > \sigma_0$ the growth rate >0; for $\sigma < \sigma_0$ the growth rate <0; and for $\sigma = \sigma_0$ growth rate =0. The simplest way to ensure this in this model is to make the growth rate be a linear function of $\sigma - \sigma_0$. It occurs the following Cauchy problem:

$$\frac{P'(t)}{P(t)} = a(\sigma - \sigma_0) \quad , a > 0 \qquad \Leftrightarrow \qquad P'(t) = a(\sigma - \sigma_0)P(t) \quad , P(t_0) = P_0 \tag{4}$$

Here a and σ_0 are constants, dependent only on the species, and σ is a parameter, dependent on the particular environment but constant for a given ecology. Integrating we obtain the solution:

$$P(t) = e^{a(\sigma-\sigma_0)(t-t_0)}P_0 \tag{5}$$

Thus the population must increase without limit, dies approaching zero as a limit, or remain constant (Figure 2).

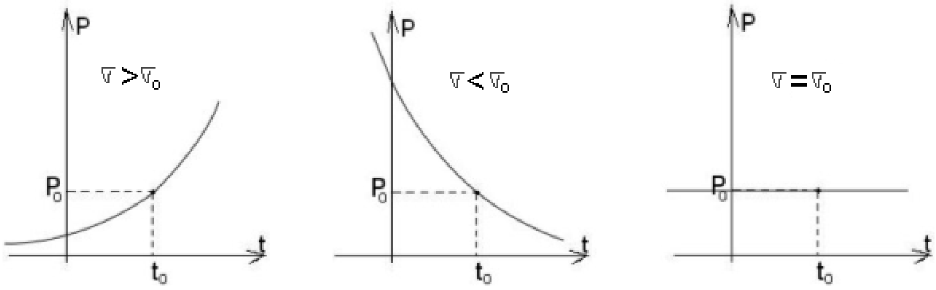

FIGURE 2. Unlimited growth model with proportional ratio

E3 Model: In reality, a population cannot increase without limit. So it is more realistic to assume that exists a certain value $\eta > 0$ called limiting population. While $P(t) < \eta$ then the growth rate >0; when the population level exceeds the value of η, so that $P(t) > \eta$ then the growth rate becomes negative; and if $P(t) = \eta$ then the growth rate is 0. We suppose in this case that the growth rate is proportional to $\eta - P(t)$:

$$\frac{P'(t)}{P(t)} = c(\eta - P(t)) \quad , P(t_0) = P_0 \quad , c > 0 \tag{6}$$

This is the equation of limited growth. Solving the Bernoulli differential equation the formula is:

$$P(t) = \frac{P_0\eta}{(\eta - P_0)e^{-c\eta(t-t_0)} + P_0} \tag{7}$$

We can note that $P(t_0) = P_0$ and $\lim\limits_{t\to\infty} P(t) = \eta$. This means that this mathematical model has a limited behavior [1,3,4] (Figure 3).

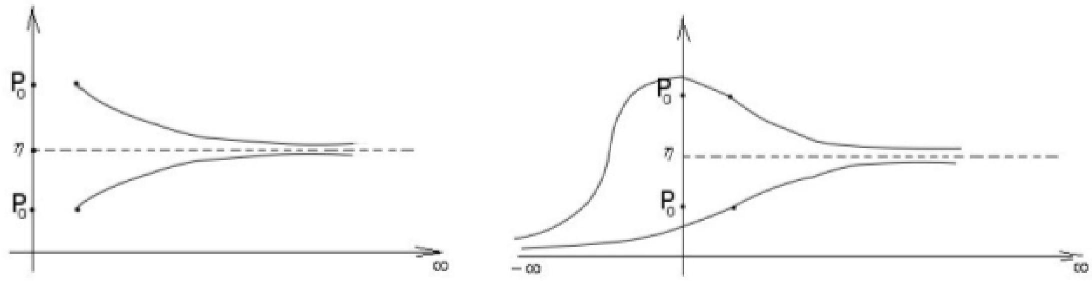

FIGURE 3. Limited growth model

3. EXAMPLE AND SOLVING

Example: Now let's study for example a population of insects, whose growth rate is proportional with 3000 - P(t). Assuming that P(0)=1000 and P(2)=1500, find the function P(t). Find the time T after that the population is double.

Solving: The equation modeling these data is

$$\frac{P'(t)}{P(t)} = k(3000 - P(t)) \tag{8}$$

which is equivalent with the following Bernoulli equation:

$$P'(t) = 3000kP(t) - kP^2(t) \tag{9}$$

We have

$$\frac{P'}{P^2} = 3000 \cdot \frac{1}{P} - k \tag{10}$$

Using the notation $\frac{1}{P} \overset{not}{=} z$ the linear equation occurs:

$$z' = -3000kz + k \tag{11}$$

Integrating the linear associated equation we have the solution

$$z(t) = Ce^{\int -3000kdz} = Ce^{-3000kt} \tag{12}$$

Then applying the Lagrange method we get

$$z(t) = u(t)e^{-3000kt} \tag{13}$$

Using the derivative of (13) and including the result in (11) occurs:

$$z' = u'e^{-3000kt} + ue^{-3000kt}(-3000k) \tag{14}$$

Associating (11) and (14) we obtain that

$$u(t) = k\int e^{3000kt}dt = \frac{k}{3000k}e^{3000kt} + C = \frac{1}{3000}e^{3000kt} + C \quad (15)$$

Including u(t) in (13) the result is

$$z(t) = \left(e^{3000kt}/3000 + C\right)e^{-3000kt} = 1/3000 + Ce^{-3000kt} \quad (16)$$

And finally the solution is

$$P(t) = 1/\left(1/3000 + Ce^{-3000kt}\right) \quad (17)$$

Using the initial conditions the values for C and k are given (C=1/1500; k=ln2/6000;). The function modeling our population growth is

$$P(t) = 3000/\left(1 + 2e^{-t\cdot \ln 2/2}\right) \quad (18)$$

For P(T)=2000 we can find the time T=4 after that the population is double.

We can observe that $\lim_{t\to\infty} P(t) = 3000$, so this is a limited growth model (Figure 4).

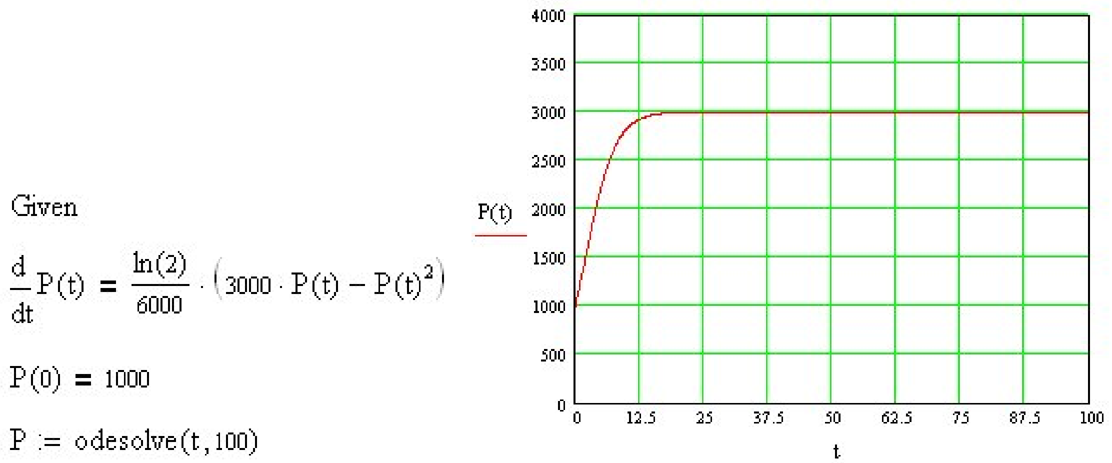

FIGURE 4. The limited growth model computed in Mathcad

4. CONCLUSIONS

In practice, the goal of differential equations is not fully met because of two reasons: the difficulty of finding the better formulas that actually solve the differential equations. We can give instead a numerical approximation of the differential equation's solution. The second reason is more dramatically in a manner of speaking: it is difficult to built a mathematical model that perfectly simulate an evolution. This is because the nature does never solve differential equations. We can only hope to make a right theoretical apparatus offering ourselves an idea on what really happened.

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