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MATHEMATICAL CONSIDERATIONS UTILIZED IN DEDUCTION OF FUNDAMENTAL EQUATION OF FLUIDS FLOW THROUGH POROUS MEDIUM

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Abstract:
For apply into another practical cases of power domain same processes and phenomena capable to extents is necessary to obtain mathematical relations extremely clears and exacts which modulating very well the reality. In this paper the authors' shown take a few examples in deduction of fundamental equation of fluids flow through porous medium. To deduce the fundamental equations we take as example:

- equation of continuity (written in global and local form);
- equation of energy transfer (global form) will be used some superior mathematical notion, respectively:

The paperwork propose to present the mode in that the superior mathematical notions (above mentioned and others too, with strong connection with these), are utilized to deduce the fundamental equations viewing the fluids flow through porous medium. Properly, there will be deduces the equation of continuity in global and local form (adapted to porous medium).

Keywords:
equation of continuity, equation of energy transfer, deduction, fluids flow, porous medium

1. INTRODUCTION

Definitions of mathematical operators in use:

a. $\nabla \overset{def}{=} \vec{i}\cdot\frac{\partial}{\partial x}+\vec{j}\cdot\frac{\partial}{\partial y}+\vec{k}\cdot\frac{\partial}{\partial z}$ (operator "nabla" or Hamilton operator)

b. $\Delta \overset{def}{=} \nabla(\nabla)=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ (operator of Laplace)

c. Material derivative for a Φ function and for an integral of volume:

$$\frac{d\Phi}{dt}=\frac{\partial\Phi}{dt}+v_i\frac{\partial\Phi}{dt}\Leftrightarrow\frac{d\Phi}{dt}=\frac{\partial\Phi}{dt}+\left(\vec{v}\cdot\nabla\right)\Phi \text{, respectively}$$

$$\frac{d}{dt}\left(\int_{V_t}\Phi dV\right)=\int_{V_t}\left[\frac{d\Phi}{dt}+\nabla\left(\Phi\vec{v}\right)\right]dV$$

d. Formulas of Gauss – Ostrogradsky:

$$\iint_S \vec{n}\cdot\vec{v}dS=\iiint_D div\vec{v}dV \overset{convention}{\Leftrightarrow} \int_S \vec{n}\cdot\vec{v}dS=\int_V div\vec{v}dV$$

2. Mathematical considerations

In the deducing of fundamental mathematical equation, viewing the fluid flowing through porous medium, will be utilized notions of superior mathematics such as:

- differential operators of I and II order (utilizing / operating modes);
- derivative of material (or substantial derivative) for a $\Phi = \Phi(\vec{r};t)$ or for a volume integral: $\int_V \Phi dV$.

2.1. Differential operators of I and II order

The "nabla " operator (or operator of Hamilton) is defined as:

Definition 1: $\nabla \overset{def}{=} \vec{i} \cdot \frac{\partial}{\partial x} + \vec{j} \cdot \frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ (that in physical point of view is a "symbol vector")

Utilizing modes of "nabla "operator

Definition 2: $\nabla f = \text{grad} f \overset{def}{=} \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ (gradient of f scalar function: vector)

Definition 3: $\nabla \vec{f} = \text{div}\vec{f} \overset{def}{=} \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$ (scalar: divergence of vector function $\vec{f}$)

Definition 4: $\nabla x\vec{f} = \text{rot}\vec{f} \overset{def}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$ (vector: rotor of vector function $\vec{f}$)

Definition 5: $\left(\vec{v} \cdot \nabla\right) \overset{def}{=} v_x \cdot \frac{\partial}{\partial x} + v_y \cdot \frac{\partial}{\partial y} + v_z \cdot \frac{\partial}{\partial z}$ (scalar produce between operator and ∇ $\vec{v}$ vector)

Definition 6: $\left(\vec{v} \cdot \nabla\right)\vec{f} \overset{def}{=} v_x \cdot \frac{\partial f}{\partial x} + v_y \cdot \frac{\partial f}{\partial y} + v_z \cdot \frac{\partial f}{\partial z}$ (the <u>material derivative</u> of a vector function $\vec{f}$)

Definition 7: $\frac{df}{dn} \overset{def}{=} \text{grad} f \cdot \vec{n} \overset{not}{=} \left(\vec{n} \cdot \nabla\right) f$ (scalar function derivative f, in direction on $\vec{n}$ versor)

Definition 8: $df \overset{def}{=} \text{grad} f \cdot \overrightarrow{dr} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ (scalar: scalar function variation f)

2.2. The operator of Laplace

Definition 9: $\nabla(\nabla) \overset{not}{=} \Delta \overset{def}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (definition of operator of Laplace)

Utilizing modes of operator of Laplace

Definition 10: $\Delta f \overset{def}{=} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (scalar: the <u>operator of Laplace</u> applied to a scalar function f)

Definition 11: $\Delta\vec{f} \overset{def}{=} \frac{\partial^2 \vec{f}}{\partial x^2} + \frac{\partial^2 \vec{f}}{\partial y^2} + \frac{\partial^2 \vec{f}}{\partial z^2}$ (vector: the operator of Laplace applied to a vector function $\vec{f}$)

Observation:

$$\Delta f \overset{not}{=} \nabla(\nabla f) \overset{def}{=} \text{div}(\text{grad} f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \overset{not}{=} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \text{ (scalar)}$$

2.3. Material derivative:

2.3.1. Material derivative for a function $\Phi = \Phi(\vec{r};t)$

Definition 12: $\frac{d\Phi}{dt} \overset{def}{=} \frac{\partial \Phi}{\partial t} + v_i \frac{\partial \Phi}{\partial t} \overset{not}{=} \frac{\partial \Phi}{\partial t} + (\vec{v} \cdot \nabla)\Phi$

where:

$$\begin{cases} \Phi = \Phi(\vec{r};t) & \text{- property associated to the fluid} \\ \frac{\partial \Phi}{\partial t} & \text{- local variation component for } \Phi \text{ property} \\ v_i = \frac{\partial x_i}{\partial t} \quad (i = 1,2,3) & \text{- components of velocity: } \vec{v}(v_x, v_y, v_z) \\ (\vec{v} \cdot \nabla)\Phi & \text{- convective component for } \Phi \text{ property} \end{cases}$$

2.3.2. Material derivative for a volume integral

Definition 13: $\frac{d}{dt}\left(\int_{V_t} \Phi dV\right) = \int_{V_t}\left[\frac{d\Phi}{dt} + \Phi(\nabla \cdot \vec{v})\right] dV$ (the first form of material derivative, for a volume integral)

Definition 14: $\frac{d}{dt}\left(\int_{V_t} \Phi dV\right) = \int_{V_t}\left[\frac{\partial \Phi}{\partial t} + \nabla(\Phi \cdot \vec{v})\right] dV$ (the second form of material derivative)

Definition 15: $\frac{d}{dt}\left(\int_{V_t} \Phi dV\right) = \frac{\partial}{\partial t}\int_{V_t} \Phi dV + \int_{S_t} \vec{n} \cdot \Phi \vec{v}\, dS$ (the third form of material derivative)

Observation: In specialty literature, the above three derivative forms of the material are knowing as "the transmission theorem" (theorem of Reynolds).

2.4. Formula's of Gauss-Ostrogradsky

2.4.1. The integral formula of Gauss-Ostrogradsky under vector form has the following mathematical expression:

$$\iint_S \vec{n} \cdot \vec{v}\, dS = \iiint_D \operatorname{div} \vec{v}\, dV \qquad (1)$$

or equivalent (using a notation convection):

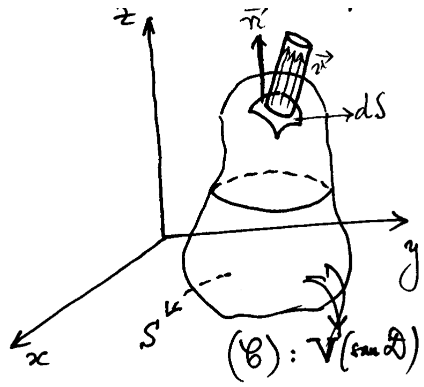
$$\int_S \vec{n} \cdot \vec{v}\, dS = \int_V \operatorname{div} \vec{v} \cdot dV \qquad (1')$$

Observation:

1. The integral formula of Gauss – Ostrogradski, given by relations (1,1') allows the following physical interpretation: "the vector flux field $\vec{v}$, that $\Phi = \iint_S \vec{n} \cdot \vec{v}\, dS$ (total flux of Φ that traverses the S closed surface) is equal to triple integral of $\vec{v}$ divergence: $\iiint_D \operatorname{div} \vec{v} dV$".

   As: $\iint_S \vec{n} \cdot \vec{v}\, dS = \iiint_D \operatorname{div} \vec{v} dV$ ("flux-divergence" formula) (2)

2. Geometrical interpretation of "flux-divergence" formula:



Notations: Oxyz – Cartesian reference points in $\mathbf{R^3}$ space
S – the closed surface of field (C)
V(or D) – the volume of the field (C)
dS – element of surface
dV – element of volume
$\Phi = \iint\limits_S \vec{n} \cdot \vec{v} dS$ flux of vector field $\vec{v}$

$\operatorname{div} \vec{v} = \nabla \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ divergence of $\vec{v}$

Discussion:
1. If $\operatorname{div}\vec{v} \rangle 0$ than: "goes out" more mass from the V volume, as the entered one (exists "positive sources")
2. If $\operatorname{div}\vec{v} \langle 0$ than: "goes in" more mass as "comes out" (exists negative sources)
3. If $\operatorname{div}\vec{v} = 0$ than: didn't exist mass losses

2.4.2. Lemma of null integral

Enunciation: " If $\Phi = \Phi(\vec{r})$ is a function that defines a continuous scalar field (or vector field) on an arbitrary domain: $D_i \subset \mathbf{R^3}$, and dV is the element of volume, than:

$$\int\limits_{D_i} \Phi(\vec{r}) dV = 0 \overset{\Rightarrow}{\Leftarrow} \Phi(\vec{r}) = 0 \text{ and inversely, (lema of null integral)} \quad (3)$$

3. Deducing the fundamental equations of fluid flows through porous medium

3.1. The equation of continuity in integral (global) and local form

3.1.1. The continuity equation in integral / local form

The stage to obtain the continuity equations are:
1. Its starts from the mass conservation principle, for than we have:
   a. the following enunciation:
   For an elementary volume of porous medium dV, the mass variation in time is $\frac{dm}{dt}$ zero.

b. $\frac{dm}{dt} = 0 \Leftrightarrow \frac{d}{dt}\left(\int\limits_{V_t} \rho m_e dV \right) = 0$ (the principle of mass conservation) (4)

2. We apply the mass derivative definition for a volume integral, the third form for relation (4):

$$\frac{d}{dt}\left(\int_{V_t}\rho m_e dV\right)=0 \Leftrightarrow \frac{\partial}{\partial t}\int_V \rho m_e dV+\int_S \vec{n}\cdot\vec{v}\rho\, dS=0 \tag{5}$$

3. For the left member of relation (5) we apply the Gauss-Ostrogradsky theorem (that realizes the transaction from the surface integral to volume integral) and we obtain:

$\frac{\partial}{\partial t}\int_V \rho m_e dV+\int_S \vec{n}\cdot\vec{v}\rho\, dS=0 \underset{\text{Ostrogradsky}}{\overset{\text{Gauss}}{\Rightarrow}} \int_V\left[\frac{\partial(\rho m_e)}{dt}+\nabla(\rho m_e\vec{u})\right]dV=0$ (the equation of continuity in integral form (global form)) (6)

4. For relation (6) we apply the null integral lemma and it will result:

$$\frac{\partial(\rho m_e)}{dt}+\nabla(\rho m_e\vec{u})=0 \tag{7}$$

(the equation of continuity for porous medium, written in local form.

3.1.2. Characterization of equation of continuity in local form

$$\frac{\partial(\rho m_e)}{dt}+\nabla(\rho m_e\vec{u})=0$$

Discussion:

Case I: If $\begin{cases}\rho\neq cst\\ m_e\neq cst\end{cases} \overset{\text{than}}{\Rightarrow} \frac{\partial(\rho m_e)}{\partial t}+\nabla(\rho\vec{v})=0$ (8)

- the equation of continuity in local form for general case

Case II: If $\begin{cases}\rho=cst\\ m_e=cst\end{cases} \Rightarrow \left.\begin{matrix}\frac{\partial(\rho m_e)}{\partial t}=0\\ \text{and } \nabla(\rho\vec{v})=\rho\nabla\vec{v}\end{matrix}\right\} \overset{\text{than}}{\Rightarrow} \left.\begin{matrix}\rho\nabla\vec{v}=0\\ \text{but } \rho\neq 0\end{matrix}\right\} \Rightarrow \nabla\vec{v}=0$ (the equation of continuity in a particular form) (9)

Observation: $\nabla\vec{v}=0 \Leftrightarrow \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial_z}=0$ (the equation of continuity)

3.2. Equation of energy transfer (written in global form)

3.2.1. Energy transfer

Enunciation: The total energy transfer through S boundary of volume V (from porous medium) is equal to dissipated energy (energy losses), as:

$$\int_S \rho\vec{n}\vec{v}\left[\frac{u^2}{2}+gz+\frac{p}{\rho}\right]dS=\int_S \rho\vec{v}\vec{f}_r dV \tag{11}$$

where:

$$\begin{cases}\int_S \rho\vec{n}\cdot\vec{v}\left[\frac{u^2}{2}+gz+\frac{p}{\rho}\right]dS=E_t\\ \int_V \rho\vec{v}\cdot\vec{f}_r dV=-\int_V \rho\lambda v^2 dV=E_d\\ \vec{f}_r=-\lambda\vec{v}\\ \vec{v}\end{cases} \tag{12}$$

- total energy: kinetically energy, position and pressure
- dissipate energy: negative energy
- fraction forces intensity
- velocity of filtration in porous medium

3.2.2. Observation

The equation of energy transfer in permanent motion regime becomes:

$$\frac{\partial}{\partial t}\int_V \rho\frac{v^2}{2m_e}dV=0 \tag{13}$$

but

$$\frac{\partial}{\partial t}\int_V \rho \frac{v^2}{2m_e} dV = -\int_S \rho \vec{n} \cdot \vec{v}\left[\frac{u^2}{2} + gz + \frac{p}{\rho}\right] dS + \int_V \rho \vec{v} \cdot \vec{f}_r dV \quad (14)$$

Than, from relations (13, 14) we can obtain:

$$\int \rho \vec{n} \cdot \vec{v}\left[\frac{u^2}{2} + gz + \frac{p}{\rho}\right] dS = \int_V \rho \vec{v} \cdot \vec{f}_r dV$$

4. Conclusion

In this paper I proposed to present the mode in that the mathematical apparatus and:

- The differential operator of I and II order: operator nabla and the operator of second order "Δ" (Laplace) together with:
- the derivative of material (for a function and for a volume integral) are useful to resolve some problems in porous medium.
- The medium formes from 3 phases:
- solid phase
- gases phase (the air)
- liquid phase (filtration fluid)

The two last phases are continuing in a net of pores, interconnected pores, through that has the fluid a circle to that I deduced the continuity equation, in two forms integral (global) and local.

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