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DOUBLE INTEGRAL INEQUALITES OF NEWTON TYPE

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ABSTRACT:
New double integral inequalities of Newton's quadrature rule are given. We give a numerical example.
Keywords:
Newton's inequality, double integral inequalities; numerical integration

1. INTRODUCTION

Let $f:[a,b]\to\mathbb{R}$, $f\in C^4([a,b])$ and $x_1,x_2\in[a,b]$ so that $x_1=\frac{2a+b}{3}$, $x_1=\frac{a+2b}{3}$ then as it is well known the relation [6] is obtained

$$\int_a^b f(x)dx=\frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right]+R \tag{1}$$

where

$$R=\int_a^b \varphi(x)f^{(4)}(x)dx \tag{2}$$

The relation gives the function:

$$\varphi(x)=\begin{cases}\frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!} & x\in[a,x_1]\\ \frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!}-\frac{9h}{8}\frac{(x-x_1)^3}{3!} & x\in[x_1,x_2]\\ \frac{(x-a)^4}{4!}-\frac{3h}{8}\frac{(x-a)^3}{3!}-\frac{9h}{8}\frac{(x-x_1)^3}{3!}-\frac{9h}{8}\frac{(x-x_2)^3}{3!} & x\in[x_2,b]\end{cases} \tag{3}$$

2. MAIN RESULTS

Under the assumptions of the quadrature formula (1) we have the next theorem:

Theorem 1

Let $f\in C^4[a,b]$. Then:

$$\frac{23\gamma_4 - 15S_3}{51840}(b-a)^5 \le \frac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x\ ) + f(b)\right] - \int_a^b f(x)dx \le \le \frac{23\Gamma_4 - 15S_3}{51840}(b-a)^5 \quad (4)$$

where $\gamma_4, \Gamma_4 \in \mathbb{R}$, $\gamma_4 \le f^{(4)}(x) \le \Gamma_4$, for any $x \in [a,b]$ and $S_3 = \frac{f'''(a) - f'''(b)}{b-a}$.

If $\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x),\ \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$ then inequalities are sharp.

Proof. From (2) integrating by parts we get :

$$\int_a^b \varphi(x) f^{(4)}(x)dx = \int_a^b f(x)dx - \frac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] \quad (5)$$

It is easy to see [2] that we get the equality:

$$\int_a^b \varphi(x)dx = -\frac{(b-a)^5}{6480} \quad (6)$$

From (5) and (6) we get the equalities:

$$\int_a^b \left[f^{(4)}(x) - \gamma_4\right]\varphi(x)dx = \int_a^b f(x)dx - \frac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] + \frac{\gamma_4}{6480}(b-a)^5 \quad (7)$$

and

$$\int_a^b \left[\Gamma_4 - f^{(4)}(x)\right]\varphi(x)dx = -\int_a^b f(x)dx + \frac{b-a}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right] - \frac{\Gamma_4}{6480}(b-a)^5 \quad (8)$$

On the other hand:

$$\int_a^b \left[f^{(4)}(x) - \gamma_4\right]\varphi(x)dx \le \max_{x\in[a,b]}\left|\varphi(x)\right| \int_a^b \left|f^{(4)}(x) - \gamma_4\right| dx \quad (9)$$

From (3) we get:

$$\max_{x\in[a,b]}\left|\varphi(x)\right| = \frac{(b-a)^4}{3456} \quad (10)$$

On the other hand the equality follows:

$$\int_a^b \left|f^{(4)}(x) - \gamma_4\right| dx = \int_a^b \left(f^{(4)}(x) - \gamma_4\right)dx = f'''(b) - f'''(a) - \gamma_4(b-a) = \left(S_3 - \gamma_4\right)(b-a) \quad (11)$$

From the relations (7), (9),(10) and (11) it follows :

$$\int_a^b f(x)dx - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \le \frac{15S_3-23\gamma_4}{51840}(b-a)^5 \quad (12)$$

the first inequality of (4).

We also have:

$$\int_a^b \left[\Gamma_4 - f^{(4)}(x)\right]\varphi(x)dx \le \max_{x\in[a,b]}|\varphi(x)|\int_a^b \left|\Gamma_4 - f^{(4)}(x)\right|dx \quad (13)$$

and

$$\int_a^b \left|\Gamma_4 - f^{(4)}(x)\right|dx = \int_a^b \left(\Gamma_4 - f^{(4)}(x)\right)dx = \Gamma_4(b-a) - f'''(a) - f'''(b) = \left(\Gamma_4 - S_3\right)(b-a) \quad (14)$$

By analogy from (8), (10),(13) and (14) we get

$$\int_a^b f(x)dx - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \ge \frac{15S_3-23\Gamma_4}{51840}(b-a)^5 \quad (15)$$

The last relation and (12) lead us to the inequality (4).

To show that inequality (4) is sharp we consider the function f given by the relation $f(x)=(x-a)^4$.

It is easy to see that the equalities $f^{(4)}(x)=24$ and $\gamma_4=\Gamma_4=24, S_3=24$ are obtained.

Calculating the three members of the inequality (4) under the given circumstances, we notice that this have the common value given by the expression $\frac{1}{270}(b-a)^5$.

Hence, we deduce that the inequality (4) is sharp.

Another relation is given by the next theorem:

Theorem 2. Under the assumptions of Theorem 1 we have:

$$\frac{7\gamma_4-15S_3}{51840}(b-a)^5 \le \int_a^b f(x)dx - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \le \\ \le \frac{7\Gamma_4-15S_3}{51840}(b-a)^5 \quad (16)$$

If $\gamma_4 = \min_{x\in[a,b]} f^{(4)}(x),\ \Gamma_4 = \max_{x\in[a,b]} f^{(4)}(x)$ then the inequalities (16) are sharp.

Proof. From (7), (9),(10) and (11) we have:

$$-\int_a^b f(x)dx + \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \le \frac{15S_3-7\gamma_4}{51840}(b-a)^5 \quad (17)$$

By analogy from (8), (13),(14) and (15) we have:

$$\int_a^b f(x)dx - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right] \le \frac{7\Gamma_4 - 15S_3}{51840}(b-a)^5 \qquad (18)$$

From (17) and (18) we will have immediately the inequalities (16).

To show that the inequalities are sharp we choose $f(x)=(x-a)^4$ and we follow the steps of the proof for Theorem 1.

3. A numerical example:

Here we consider the integral $\int_0^1 e^{x^2}dx$. We now compare the result obtained in Theorem 2 with the usual Peano error bound [4]:

$$\left|\int_a^b \varphi(x)f^{(4)}(x)dx\right| = \left|\int_a^b f(x)dx - \frac{b-a}{8}\left[f(a)+3f(x_1)+3f(x_2)+f(b)\right]\right| \le$$
$$\le \left\|f^{(4)}\right\|_\infty \int_a^b \left|\varphi^{(4)}(x)\right|dx = \frac{(b-a)^4}{6480}\left\|f^{(4)}\right\|_\infty \qquad (19)$$

We have $f(x)=e^{x^2}$, S_3=20e, $\gamma_4 = 12$, $\Gamma_4 = 76e$ (on the interval [0,1]). From (16) we have:

$$\int_a^b e^{x^2}dx - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right] \le \frac{29e}{6480} \qquad (20)$$

From (19) we get:

$$\int_a^b e^{x^2}dx - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right] \le \frac{76e}{6480} \qquad (21)$$

It is obvious (20) is better than (26).

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