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THE INTERACTIVE GEOMETRY SOFTWARE *CINDERELLA*

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ABSTRACT
This paper is about the foundations of Dynamic Geometry with respect to mathematics, computer science and education. Up to today there was no software which could handle dynamic manipulation of two-dimensional constructions in a mathematically satisfying way. Only in the pure mathematics the principle of continuity of Poncelet, stating that any mathematical property must remain valid even if it is not visible, has found its place; geometry, which was its birthplace, went on untouched. Even worse, nowadays the domains of validity of constructions are examined, which are not caused by mathematics, but by specific implementations within the software used.
KEYWORDS
Graphical interface, educational software, dynamic geometry

1. INTRODUCTION

Cinderella is a software package for doing geometry on a computer (1). In a way it replaces pencil, paper, ruler and compass with equivalent computer tools. With the mouse you "draw" in a window on screen, i.e. you place points, connect them by lines, erect perpendiculars, etc.

This functionality is not very exciting, but already useful if you want to do exact constructions. You can be sure that you exactly – only restricted by the floating point accuracy of the computer – hit an intersection point of two lines, or draw an exact parallel to some other line. Also some constructions that are tedious to do by hand are easily done with the computer, for example inversions at a circle (or conic). The additional possibility of rescaling a figure can help you if the construction you are doing exceeds the limits of the window.

The important distinction of Cinderella to other drawing software – Corel Draw, Microsoft Word or others – is that it keeps track of your construction steps and is able to re-do them, even for a different placement of the base elements. This happens instantly and interactive, so you can pick a point and drag it to some other position, and the rest of construction is updated during the move.

This looks and feels like having build the construction from matter. This feature, which is characteristic for interactive geometry software, is very useful while you do the construction: You can adjust it in order to avoid crowded parts of a drawing, to make it look nicer, or to have access to elements that lie outside the drawing region. But is even more useful when you have finished the construction, since you can explore the geometric properties of it. "What happens if?" is a question that can be answered with dynamic geometry software, and, even better, you can build intuition and feeling for geometry using the drag-mode of interactive geometry software.

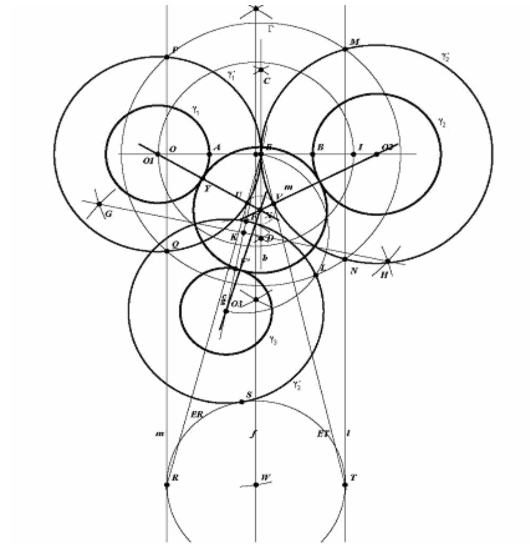

Figure 1. An example for a construction that is easy with *Cinderella*, but hard to do with ruler and compass on a sheet of paper

Cinderella supports points, lines, circles, arbitrary conics, segments, polygons, measured distances and angles as well as loci. Among the basic constructions are intersections of lines and conics, parallels and perpendiculars, angular bisectors, circles defined by their centers and a point on the circle or by three points, and conics defined by five points.

Besides the basic characteristic of interactive geometry software, the ability to move points and lines while the constructions is immediately updated; Cinderella has several unique traits, which we want to emphasize. Most of these have only been possible due to the mathematical foundation of Cinderella – we just note this to stress the importance of a profound mathematical background.

With Cinderella, you can view a single construction in different geometrical interpretations at the same time. Internally, an abstract model of the construction is maintained, which can be displayed through one or more windows – or *view ports*, as we call them. These come in many different flavors, and each of it has its own strengths.

2. EUCLIDEAN PLANE

The usual view port, which is also the default view port at startup, is the *Euclidean plane* (3). Of course, this is not really the mathematical Euclidean plane, but computer display version of it. This is the default view port, since most people are used to it and want to work in it. The pixels on the screen are mapped to a rectangular part of the Euclidean plane embedding of the projective plane, or vice-versa. Lines look like lines, circles look like circles.

The construction shows an offset curve of a parabola, which is a challenge for every CAD system. The window in the lower right of the screen shows the two internal cusps of the offset curve, the window in the upper right shows a detail view of the right cusp. You can choose the section of the Euclidean plane you want to see by zooming in or out and by translating. Just imagine you have a variable-sized rectangle you can put on the Euclidean plane, and everything inside that rectangle will be displayed on screen.

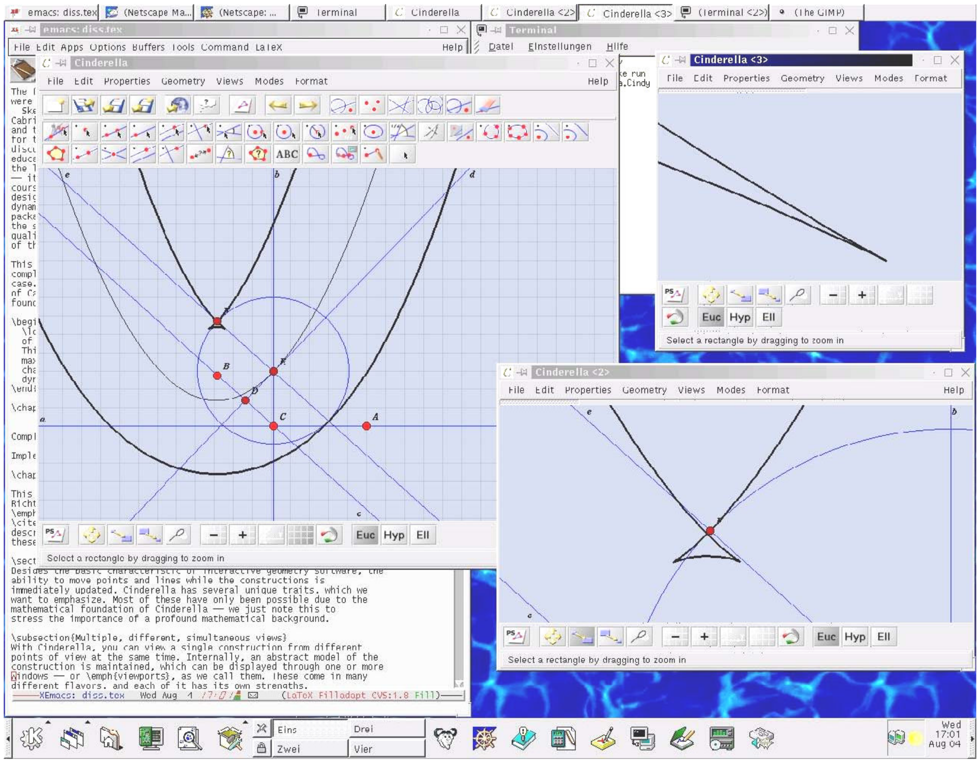

Figure 2. A Cinderella session with three different Euclidian view ports

Even if you only had this single kind of view port, it would make sense to have multiple, simultaneous copies. A possible scenario: You want to explore a locus – an algebraic curve created as the trace of an object – that has an interesting bend. You want to zoom into the construction, since you cannot see whether there is a cusp at that position or not. At the same time you want to have a global overview over the construction, because you want to move points to see how the locus changes at that position. With Cinderella you can open a second Euclidean view port to have several views at the same.

3. SPHERICAL PROJECTION

Inside Cinderella the Euclidean plane is augmented by a "line at infinity," which embeds the plane into a space called the projective plane. This projective plane can be visualized as a double-cover of the unit sphere in three dimensional space: Points are mapped to pairs of antipodal points on the sphere, and lines are mapped to great-circles around the sphere. The map is given by a central projection of the plane located at z 1 in 3-space through the origin onto the unit sphere. The "north pole" of the sphere touches the plane. The equator of the sphere does not correspond to any point of the Euclidean plane, instead we can find the "points at infinity" there.

We can use the spherical projection view port to get a better understanding of the concept of infinity. Most people have heard that parallel lines meet at infinity, but they could never verify this by experience, or even imagine it. In Cinderella you can open a spherical view and move a point straight to the equator, and you can see that all lines meeting this point become parallel in the Euclidean view port. The position on the equator determines the direction of the parallel bundle (Figure 3).

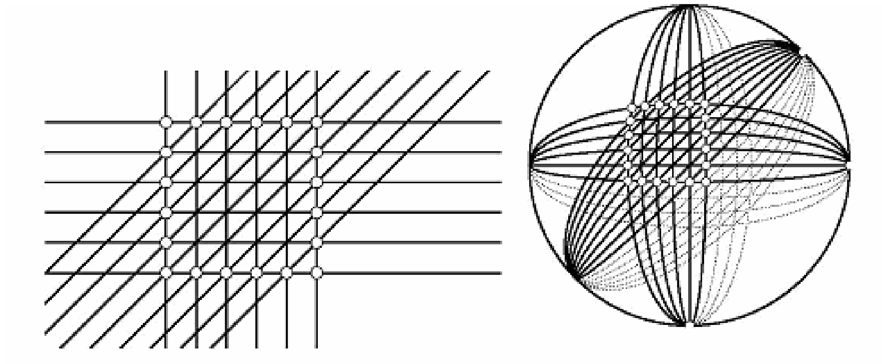

Figure 3. Three parallel bundles, on the left in a Euclidian viewport, on the right in the sperical projection

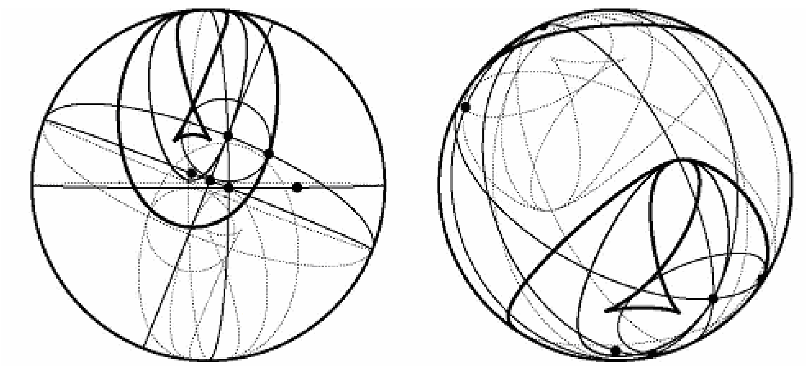

Figure 4. The left picture shows the offset parabola in a spherical port with the line at infinity on the boundary of the circle and the right picture shows the same situation after rotating the sphere

While it is very instructive to explain infinity to somebody using spherical view ports, its applications go far beyond. For example, you can study the behavior of a locus curve at infinity: Take the offset parabola of figure 2. In the spherical port you can see that the parabola touches the line at infinity, and the offset curve touches at the same point (Figure 4). When you animate the construction you can see how the offset point *C* changes from one side of the parabola to the other.

4. POINCARÉ DISC MODEL OF HYPERBOLIC GEOMETRY

Hyperbolic geometry is probably the most famous non-Euclidean geometry. Several drawings of M.C. Escher (2) show tilings of a circle that become finer and finer to the boundary. These drawings use the Poincar´e-model of the hyperbolic plane: The fundamental object, which plays the role of infinity, is the outer circle. Since it is at infinity, we cannot reach it by making a finite number of steps of some constant distance.

This is shown in figure 5: You are trapped inside the circle if you allow only finite hyperbolic distances. The hyperbolic plane is a wonderful playground for geometric explorations. You can check, for example, how your favorite theorem looks like in the hyperbolic plane. If you can do hyperbolic measurements, you can check which constructions of the Euclidean plane are still valid, and which of them fail. Did you know that two hyperbolic circles may have 4 intersections? What does this mean for constructions that use circle/circle intersections? These and other questions take you to the roots of geometry.

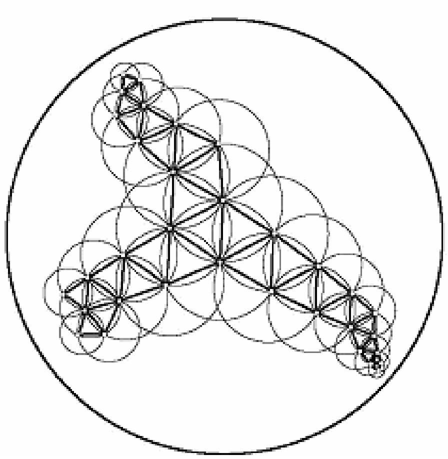

Figure 5. Try to reach the boundary of the Poincaré view by making hyperbolic unit steeps. The circle is to the hyperbolic plane what the line at infinity is to the Euclidean plane

Other geometry software emulates the Poincaré disc model with construction macros for circles in a Euclidean view. A big advantage of a built-in view is its speed, its completeness and its correctness. Especially completeness is hard to achieve with macro-based approaches, since this would mean to provide macros that work on loci – which no Dynamic Geometry software so far can do. The correctness is just a matter of correct macro constructions, but this seems to be not as easy as one might think: For example, the macro should still work for base elements that lie outside the fundamental conic, "behind infinity."

5. COMPLEX NUMBERS

Cinderella uses complex numbers for all coordinates and values. Intermediate construction elements may become complex, but they usually do not disappear (this might happen in degenerate positions only). The construction is carried out until the end with these imaginary elements, and if a result is again real, then it will be displayed.

A particularly nice example of why this is useful is the *radical axis* of two circles. The radical axis is a line that is perpendicular to the line connecting the midpoints of the two circles, and which lies between the midpoints cutting their joining segment *s* into two parts at a certain ratio defined by the distance and the radii of the circles. For equally-sized circles the radical axis happens to be the perpendicular bisector of the segment *s*.

When the two circles intersect, then the radical axis meets the two intersection points. This means that we can define it also as the join of these two intersections. Usually, this construction is considered valid only if the circles intersect, but algebraically the above is also true when we move the two circles apart. The two intersections did not vanish, they became complex. They are defined by the two solutions of a polynomial of degree two with real coefficients, thus they are complex conjugates and their join is again a real line with real-valued coordinates.

This feature is sometimes confusing for the user, especially in a classroom environment.

6. CAYLEY-KLEIN GEOMETRIES

In the view port section we already mentioned hyperbolic measurements. Cinderella offers three different measurements, the usual Euclidean distances and angles, elliptic measurements, which occur when you want to measure distances and angles on a sphere, and hyperbolic measurements.

You can use any of the measurements in any of the view port, there is no direct connection between displaying and measuring. Of course, it often makes more sense to use the hyperbolic measurement in the hyperbolic view port, etc. For example, the Poincaré disc model is coherent with hyperbolic angle measurements, the Euclidean angles you measure between the circular arcs are the same as the hyperbolic angles between the lines.

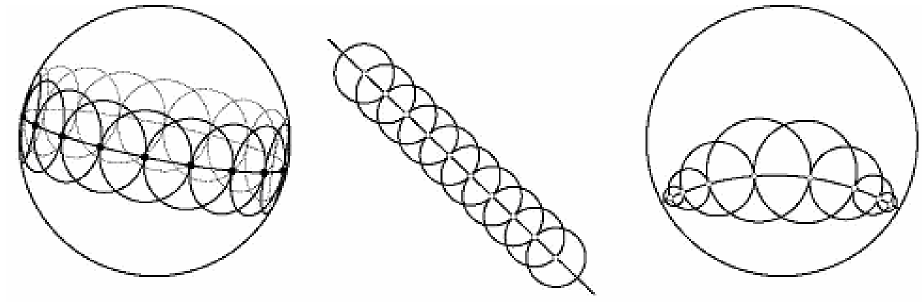

Figure 6. Circles in the three types of geometry that *Cinderella* supports

7. AUTOMATED THEOREM CHECKING

Based on the continuity of elements a randomized theorem checker has been built into Cinderella (4). The theorem checker is surveying all the construction steps done by the user and reports all non-trivial incidence theorems it finds (Figure 7). It is also used for the correctness checking of interactive exercises, see below, and to keep the internal data structures consistent.

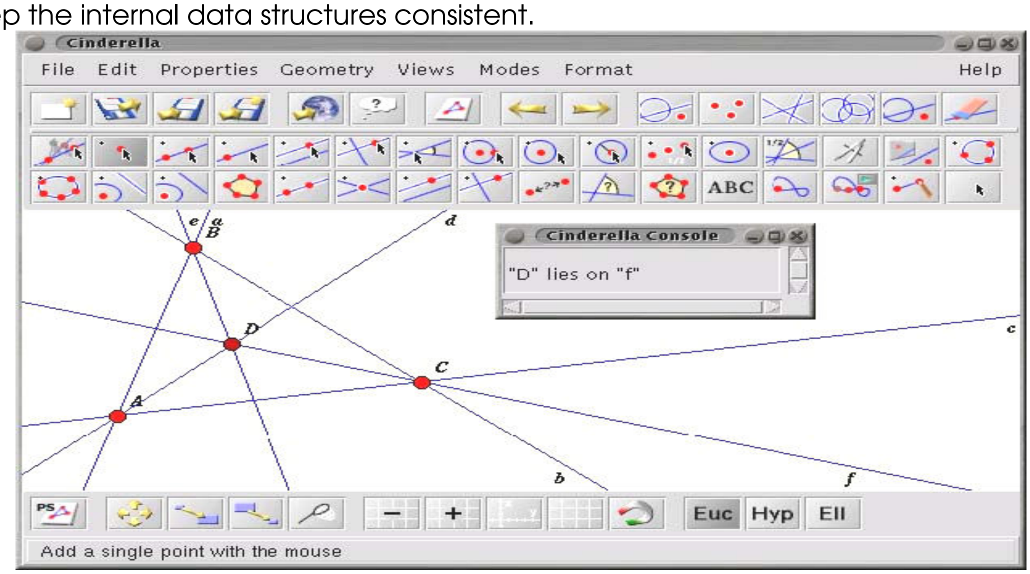

Figure 7. The theorem checker at work

8. INTERACTIVE EXERCISES

The next level of interactive geometry is reached by combining the powerful theorem checking engine of Cinderella with the web export. The results are interactive geometry exercises with automatic solution checking done by the computer.

Suppose that in a sequence of geometry lessons you have taught how to do basic constructions like angular bisector or midpoint using ruler and compass only, and now you want to check whether the students can transfer these constructions to other situations. You design an assignment that combines some of the basic constructions. This approach has several drawbacks: It is very much work for you as a teacher to check all the different solutions the students offer. Good students might come up with "better" constructions as the one you had in mind, and you have to find out whether it is really a valid solution or whether it fails for some situations. Inexperienced students might come up with no solution at all, because they got stuck after the first few steps.

The interactive exercises created with Cinderella attack these problems by offering an automatic solution checking based on the built-in theorem checking, and an automatic hinting mechanism that guides students to the next step while still not restricting them to a particular construction sequences. This flexibility ensures the greatest freedom possible for the students while still helping them not to get lost.

It is not easy to design a good exercise, but since the exercises can be accessed using the Internet it is only a matter of month to create a database of high-quality exercises in a joint effort of teacher's nation-or world-wide.

9. CONCLUSIONS

The Cinderella project was always driven by mathematics, and not by mathematics education. We are still sure that it does not make sense to adjust geometry in a way that it complies with the curricula, but it must be the curricula that are adjusted to match geometry. It is our strong believe that the principle of continuity is important for a proper understanding of geometry, and that the new possibilities of a software like Cinderella – we mean the mathematical possibilities, not the Internet support – are a chance to use geometry for teaching again. The trend should be to go back to the roots of modern geometry, and use this wonderful part of mathematics with help of the computer to teach better mathematics.

Of course, this a very personal view, and there is no empirical evidence that mathematically founded geometry software does really make a difference to other geometry software. This is a field of research for the future; but it will be hard to do this research because it needs the rethinking of geometry and math education as a whole. Again, it does not make sense to put mathematical founded geometry software into the corset of a curriculum that has been built over decades and could not be focused on the new possibilities of Dynamic Geometry – if we can overcome this, we will have a chance to go new ways in education.

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