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UPON THE STABILITY OF A LINEAR VECTOR DIFFERENTIAL EQUATION OF FIRST ORDER

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ABSTRACT:
Most of the theories examining a stability problem of the zero solution are based on the Lyapunov stability and instability theorems. This study presents theoretical considerations upon a method to observe the behaviour of the zero solution of a linear vector differential equation by studying the sign of the eigenvalues for the system matrix. A numerical example is considered and analyzed.
KEYWORDS:
stability, instability, asymptotical stability, linear differential equation

1. INTRODUCTION

A problem that has been oftentimes addressed in the stability theory (1, 2, 3) is the study of the stability of an exact solution for a given differential equation. The Section 2 presents how this problem can be reduced to the study of the zero-solution of that differential equation. For this aim it is necessary that the differential equations shown as vectors, having the form $x' = F(t,x)$, to be defined on cylinders like R_+xG, where G represents a domain of R^n, conventional noted X, as appears below. A numerical example is analyzed in Section 3 and Section 4 concludes the paper.

2. METHODOLOGY

In the case under consideration, we shall assume that the origin 0 of X belongs to G, for the reason that the differential equation must admit the zero-solution x = 0.

An efficient method to find if the zero-solution of a linear vector differential equation is stable or unstable is to study the sign of the eigenvalues of the system matix.

Let $t \rightarrow A(t) : J \subseteq R \rightarrow M(n, C)$ be a continuous function. The differential equation

$$x' = Ax \tag{1}$$

where the unknown function $t \rightarrow x(t)$ is a vector function $x : J \rightarrow X$, is called a linear vector equation of first order or differential homogeneous system.

Let assume that $(t_0, x^0) \in J \times X$, then the Cauchy problem

$$x' = Ax \ , \ x(t_0) = x^0 \tag{2}$$

can be associate with the differential equation (1).

PROPOSITION 1 (*that defines the structure for the solutions of the equation*)

The function $t \rightarrow x(t)$ represents a solution for the equation (1) *if and only if*

$$x(t) = e^{\lambda_1 t} p_1(t) + e^{\lambda_2 t} p_2(t) + ... + e^{\lambda_k t} p_k(t)$$

for any $t \in R$, where $p_j : R \to X$, $j = 1, 2...k$ are polynomial vector functions of grade less than
$n_j - 1$ and $x^j(t) = e^{\lambda_j t} p_j(t)$ are solutions for the equation (1).
Proof:

Let $t \to x(t)$ be a solution of the equation (1). We denote $x^0 = x(0)$. Due to this, the vector x^0 has a unique representation as $x^0 = x^0_1 + x^0_2 + ... + x^0_k$. For x(t) is the unique solution of the Cauchy problem (2), we have that $x(t) = e^{tA}x^0 = \sum_{j=1}^{k} e^{tA}x^0{}_j$, where $x_j(t) = e^{tA}x^0{}_j$ are obviously solutions of the equation (1).

Let $y_j(t) = e^{-\lambda_j} x_j(t) = e^{t(A-\lambda_j I)}x^0{}_j$. Hence, by successive differentiation we obtain $\frac{d^{n_j}}{dt^{n_j}} y_j(t) = (A - \lambda_j I)^{n_j} y_j(t) = 0$. This means that $y_j(t)$ is a polynomial vector function of a grade less than $n_j - 1$. This will be denoted by $p_j(t)$, hence the solution x(t) can be written in the form asked by the proposition.

Conversely, it is known that the set of solutions of equation (1) is a linear space and the formula given by the discussed proposition is a solution of equation (1).

The considered X space has exactly one decomposition in directly sum of three liniar subspaces, as follows.

We denote by $\sigma(A)$ the spectrum of matrix A, such that

$$\sigma_-(A) = \{ \lambda \in \sigma (A) \mid Re(\lambda) < 0 \} \quad (3)$$

$$\sigma_0(A) = \{ \lambda \in \sigma (A) \mid Re(\lambda) = 0 \} \quad (4)$$

$$\sigma_+(A) = \{ \lambda \in \sigma (A) \mid Re(\lambda) > 0 \} \quad (5)$$

We denote, as well, by

$X_- = \oplus \ker(A - \lambda_j I)^{n_j}$, for $\lambda \in \sigma_-(A)$ the stable subspace
$X_0 = \oplus \ker(A - \lambda_j I)^{n_j}$, for $\lambda \in \sigma_0(A)$ the central subspace
$X_+ = \oplus \ker(A - \lambda_j I)^{n_j}$, for $\lambda \in \sigma_+(A)$ the unstable subspace

The relation $X = X_- \oplus X_0 \oplus X_+$ exists in consequence.

THE STABILITY THEOREM FOR LINIAR VECTOR EQUATION OF FIRST ORDER

(a) If $\sigma_+(A) \neq \varnothing$, then the solution $x = 0$ of equation (1) *is unstable.*
(b) If $\sigma_+(A) = \varnothing$, then:
(b1) If $\sigma(A) = \sigma_-(A)$, then the solution $x = 0$ of equation (1) *is asymptotically stable.*
(b2) If $\sigma_0(A) \neq \varnothing$ and the multiplication order of at least one eigenvalue of $\sigma_0(A)$ *is greater than 1, then the solution $x = 0$ of equation* (1) *is unstable. If the multiplication order for all eigenvalues of* $\sigma_0(A)$ *is equal to 1, then the solution $x = 0$ of equation* (1) *is stable but it is not asymptotically stable.*
Proof:
(a) If there exists λ_j such that $Re(\lambda_j)>0$, it follows that

$$| e^{\lambda_j} x^0_j || = e^{Re(\lambda_j) t} || x^0_j || \to \infty, \quad \text{for } t \to \infty$$

Since $x_j(t) = e^{\lambda_j} x^0_j$ is a solution of equation (1), for any eigenvector x^0_j of matrix A, proper of the eigenvalue λ_j, it may be noted that there exists unbounded solutions of the equation (1), thus the zero-solution is unstable.
(b1) Considering that $Re(\lambda_j)<0$, for every $\lambda_j \in \sigma(A)$, based on the solution representation formula given by Proposition 1, it follows that

$$\left\|x(t;t_0,x^0)\right\| = \left\|\sum_{j=1}^{k} e^{\lambda_j} p_j(t)\right\| \le \sum_{j=1}^{k}\left\|e^{\lambda_j} p_j(t)\right\| = \sum_{j=1}^{k} e^{\mathrm{Re}(\lambda_j)t}\left\|p_j(t)\right\| \xrightarrow{t\to\infty} 0 \tag{6}$$

for every input data $t_0 \in R$ and $x^0 \in X$. Thus the zero-solution of equation (1) is asymptotically stable.

(b2) Every solution of equation (1) has the form x(t) = $x_-(t) + x_0(t)$, where $x_-(t) \in X_-$ and $x_0(t) \in X_0$. The result $x_-(t) \to 0$ pentru $t\to\infty$ follows from the (b1) proof. It is relevant to remark that

$$x_0(t) = \sum_{\lambda\in\sigma_0(A)} e^{i\mathrm{Im}(\lambda)t} p_\lambda(t) = \sum_{\lambda\in\sigma_0(A)} x_0^\lambda(t) \tag{7}$$

where p_λ are polynomials of n_λ -1 grade (where n_λ is the multiplication order for the eigenvalue λ). For those eigenvalues $\lambda \in \sigma_0(A)$ having a multiplication order equal to 1, we obtain that $p_\lambda(t) = c_\lambda \in X$ and $\left\|x_0^\lambda(t)\right\| = \left\|e^{i\mathrm{Im}(\lambda)t} c_\lambda\right\| = \left\|c_\lambda\right\|$, thus $x_0^\lambda(t)$ are bounded.

For those eigenvalues $\lambda \in \sigma_0(A)$ having a multiplication order greater than 1, observe that

$$\left\|x_0^\lambda(t)\right\| = \left\|e^{i\mathrm{Im}(\lambda)t} p_\lambda(t)\right\| = \left\|p_\lambda(t)\right\| = \left\|c_{n_\lambda-1} t^{n_\lambda-1} + \ldots\right\| \xrightarrow{t\to\infty} \infty \tag{8}$$

thus $x_0^\lambda(t)$ are unbounded.

This considerations proves that if there exists some eigenvalues $\lambda \in \sigma_0(A)$ having the multiplication order greater than 1, the solutions x(t) are unbounded for any initial value problem, thus the solution x = 0 is unstable (4, 5).

In case that all the eihenvalues $\lambda \in \sigma_0(A)$ have the multiplication order equal to 1, we infer that the solutions x(t) of the equation (1) are bounded and, in the same time, the zero-solution of equation (1) is stable, but it is not asymptotically stable for the reason that for every $\delta > 0$ it can be $x(t) = \delta e^{i\mathrm{Im}(\lambda)t} c_\lambda$ a solution of equation (1) (where $\lambda \in \sigma_0(A)$, $\|x(0)\| = \delta$ and c_λ is an eigenvector of the eigenvalue λ). For this solution we can have $\left\|x(t)\right\| = \left\|\delta e^{iIM(\lambda)t} c_\lambda\right\| = \delta\left\|c_\lambda\right\| = \delta$, thus $\lim_{t\to\infty}\left\|x(t)\right\| \ne 0$. The theorem is proved.

3. EXAMPLE AND SOLVING

Consider the equation

$$x' = Ax, \quad x = (x_1, x_2), \quad A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \tag{9}$$

The eigenvalues of A are $\lambda = 1 + i$, $\overline{\lambda} = 1 - i$.

A complex eigenvector belonging to 1+i is found by solving the equation

$$\left(A - (1+i)\right)w = 0 \tag{10}$$

for $w \in C^2$,

$$\begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \tag{11}$$

or

$$\begin{aligned} (-1-i)w_1 - 2w_2 &= 0 \\ w_1 + (1-i)w_2 &= 0 \end{aligned} \tag{12}$$

The first equation is equivalent to the second, as is seen by multiplying the second by (-1-i). From the second equation we see that the solutions are all complex and multiples of any nonzero complex vector w such that $w_1 = (-1+i)w_2$. For example $w_1 = 1+i$, $w_2 = -i$. Thus

$$w = (1+i,-i) = (1,0) + i(1,-1) = u + iv \tag{13}$$

is a complex eigenvector belonging to 1+i .

Choosing the new basis $\{v,u\}$ for $R^2 \subset C^2$, with v = (1, -1), u = (1, 0), to find new coordinates y_1, y_2 corresponding to this new basis, note that any x can be written

$$x = x_1(1,0) + x_2(0,1) = y_1 v + y_2 u = y_1(1,-1) + y_2(1,0) \tag{14}$$

Thus

$$\left.\begin{aligned} x_1 &= y_1 + y_2 \\ x_2 &= -y_1 \end{aligned}\right\} \quad \text{or} \quad x = Py\,, \quad P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \tag{15}$$

The new coordinates are given by

$$\left.\begin{aligned} y_1 &= -x_2 \\ y_2 &= x_1 + x_2 \end{aligned}\right\} \quad \text{or} \quad y = P^{-1}x\,, \quad P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \tag{16}$$

The matrix of A in the y-coordinates is

$$P^{-1}AP = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = B \tag{17}$$

Thus, the differential equation

$$\frac{dx}{dt} = Ax \tag{18}$$

on R^2, having the form

$$\frac{dy}{dt} = By \tag{19}$$

in the y-coordinates, can be solved as

$$\begin{aligned} y_1(t) &= ue^t \cos t - ve^t \sin t, \\ y_2(t) &= ue^t \sin t + ve^t \cos t \end{aligned} \tag{20}$$

The original equation has its general solution

$$\begin{aligned} x_1(t) &= (u+v)e^t \cos t + (u-v)e^t \sin t, \\ x_2(t) &= -ue^t \cos t + ve^t \sin t \end{aligned} \tag{21}$$

4. CONCLUSIONS

Most of the theories examining a stability problem of the zero solution of a linear differential equation are based on the Lyapunov stability and instability theorems. As has been shown above, this method is an easy way to get the behaviour of the zero solution only by studying the sign of the eigenvalues for the system matrix. The theorem exposed is very useful in the stability theory of first approximation.

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