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COMPUTATIONAL SIMULATION USING SERIES EXPANSION FOR BOUNDARY-LAYER MOTION

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ABSTRACT:
This paper introduces a computational method for solving the second order nonlinear differential equation with boundary condition and an integral condition, which describes the parameters of motion along a flat plate. We describe the theoretical arguments that lead to the nonlinear differential equation and then we suggest a computational method to approximate the solution with a series expansion which satisfies the nonlinear differential equation with its boundary condition and the integral condition. The results we obtained by simulating the approximate and exact values of the boundary layer parameters for the case of an incompressible flow along a flat plate are presented comparatively. Calculations have been done in MathCad.

KEYWORDS:
Boundary-layer motion, Computational fluid dynamics

1. INTRODUCTION

The theory describing the motion of a perfect fluid is mathematically very far deveolped and supplies in many cases a satisfactory description of real motions, such as the motion of surface waves or the formation of liquid jets in air. For example, the motion of a frictionless and incompressible fluid along a flat plate may describe the motion of surface winds. This paper introduces a computational method for solving the second order nonlinear differential equation with boundary condition and an integral condition, which describes the parameters of motion along a flat plate. We are going to give in Section 2 the theoretical arguments that lead to the nonlinear differential equation. We propose a computational method to approximate the solution with a series expansion which satisfies the nonlinear differential equation with its boundary condition and the integral condition. In order to obtain an optimal function corresponding to given conditions, we elaborated a program under MathCad, as it will be shown in Section 3. The series expansion we obtained can be compared to the exact solution of motion in Section 4 that presents an analysis of the results we obtained, and Section 5 includes the conclusions of the paper.

2. METHODOLOGY: DESCRIPTION OF THE ANALYTICAL METHOD

The existence of tangential stresses and the condition of no slip near solid walls constitute the essential differences between a perfect and a real fluid. As describes

H. Schlichting in (5), the motion of fluids of small viscosity such as water and air agrees very well with that of a perfect fluid, because in most cases the shearing stresses are very small.

N.I. Akatnov (1) and M.B. Glauert (3) succeeded in calculation of horizontal fluid velocity.

We consider a flat plate at $y = 0$ with a stream with constant speed parallel to the plate, x is the horizontal coordinate and y is the vertical coordinate. u and v are, respectively, the horizontal and vertical fluid velocities. The pressure does not vary in the y direction, so the pressure is constant across the boundary layer and its gradient is given by the pressure gradient outside the boundary layer. For this problem the governing equations for the fluid motion are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

$$\mu = \text{const.} \qquad \rho = \text{const.} \tag{3}$$

where μ and υ are the dynamic and kinematic viscosity, and ρ is the density ($\mu = \upsilon\rho$) .

Due to the viscosity we have the no slip condition at the plate. Furthermore, at the fluid surface there is no flow across it, which implies the boundary condition

$$\begin{aligned} &u=v=0 \quad for \quad y=0 \\ &u=0 \qquad\ \ for \quad y=\infty \end{aligned} \tag{4}$$

We consider the stream function ψ related to the velocities u and v according to the equations

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x} \tag{5}$$

We would like to find a change of variables which allows us to perform the reduction from (x, y) to von Mises (4) variables (ξ, ψ):

$$\begin{aligned} &\xi = x \\ &\psi=\psi(x,y) \end{aligned} \tag{6}$$

The fluid motion equation (1) changes this form such as

$$\frac{\partial u}{\partial\xi}=\upsilon\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right) \tag{7}$$

The boundary condition (4) lead to the new boundary condition

$$\begin{aligned} &u(\xi,\psi)=0, \quad for\ \psi=0, \\ &u(\xi,\psi)=0, \quad for\ \psi=\psi_\infty\neq\infty, \end{aligned} \tag{8}$$

where the plate is considered to be the stream $\psi = 0$. The value ψ_∞ referes to the volumetric flow rate, which is the volume of fluid which passes through a given section ξ = constant per unit time, described by

$$Q=\int_0^\infty \rho u dy=\rho\int_0^\infty u dy=\rho\psi\Big|_0^\infty=\rho\psi_\infty \tag{9}$$

In order to obtain the function u(ξ, ψ), the equation (7) with the boundary condition (8) has to be connected with an integral condition. While the boundary conditions reflect the studied phenomena according to its known external behavior, the integral conditions ensure non-null solutions, sometimes they lead to some ordinary differential equations and they are helpful in finding needed constants (2).

By multiplication of (7) with $\psi d\psi$, then integrate with respect to ψ from $\psi = 0$ to $\psi = \psi_\infty$, then with respect to ξ from $\xi = 0$ to $\xi = \xi$, an integral condition of form

$$\int_0^{\psi_\infty} u\psi\, d\psi = E_0 \tag{10}$$

occures, where E_0 is a given constant.

With the function changing

$$u(\xi,\eta) = \sqrt{\frac{E_0}{\upsilon\xi}}\varphi(\eta) \tag{11}$$

where

$$\eta = (E_0\upsilon\xi)^{-\frac{1}{4}}\psi \tag{12}$$

we reduced the problem that we have now to find the function $\varphi(\eta)$ which has to satisfy the differential equation

$$2(\varphi^2)'' + \eta\varphi' + 2\varphi = 0 \tag{13}$$

with boundary condition:

$$\begin{aligned} \varphi(\eta) &= 0, \quad for\ \eta = 0 \\ \varphi(\eta) &= 0, \quad for\ \eta = \eta_\infty \end{aligned} \tag{14}$$

and the integral condition:

$$\int_0^{\eta_\infty} \varphi(\eta)\eta\, d\eta = 1 \tag{15}$$

The analitycal method to find the solution of (13) by giving exact solutions and calculate their Wronskian, leads to

$$\varphi(\eta) = \frac{1}{6}\eta_\infty^2\left(\sqrt{\frac{\eta}{\eta_\infty}} - \frac{\eta^2}{\eta_\infty^2}\right) \tag{16}$$

Where $$\eta_\infty = \sqrt[4]{40} \approx 2{,}515 \tag{17}$$

The horizontal fluid velocity has the form

$$u(x,\eta) = \frac{1}{6}\eta_\infty^2\sqrt{\frac{E_0}{\upsilon\xi}}\left(\sqrt{\frac{\eta}{\eta_\infty}} - \frac{\eta^2}{\eta_\infty^2}\right) \tag{18}$$

Where $$\begin{aligned} \xi &= x \\ \eta &= \psi(E_0\upsilon x)^{-\frac{1}{4}} \end{aligned} \tag{19}$$

We propose a method to approximate the solution by giving a series expansion which satisfies the differential equation with its boundary condition and the integral condition.

Let the approximate solution be of form

$$f(\eta) = a + b\cdot(\frac{\eta}{\eta_\infty}) + c\cdot(\frac{\eta}{\eta_\infty})^2 + d\cdot(\frac{\eta}{\eta_\infty})^3 + \beta\cdot\sqrt{\frac{\eta}{\eta_\infty}} + \lambda\cdot\ln\left(1+\frac{\eta}{\eta_\infty}\right) \tag{20}$$

where $a = 0$ and $b + c + d + \beta + \lambda\ln 2 = 0$ from (14).

3. DESCRIPTION OF THE COMPUTATIONAL METHOD

The function that approximates not only the differential equation with given boundary condition but the integral condition, have been determined with the help of the MathCad program we are giving hereinafter. Constants b,c,d, β, λ depend on the fluid velocity function.

$ORIGIN \equiv 1 \qquad TOL \equiv 0.01$

$\eta init := 0 \quad nr := 3 \quad k := 1..nr \quad \eta\inf v_k := 2.5 + (k-1)\cdot 0.015 \quad bv_k := -0.015 + (k-1)\cdot 0.015$

$cv_k := -1.069 + (k-1)\cdot 0.015 \quad dv_k := -0.012 + (k-1)\cdot 0.012 \quad \beta v_k := -1.054 + (k-1)\cdot 2.108$

$$\eta\inf v = \begin{pmatrix} 2.5 \\ 2.515 \\ 2.53 \end{pmatrix} \quad bv = \begin{pmatrix} -0.015 \\ 0 \\ 0.015 \end{pmatrix} \quad cv = \begin{pmatrix} -1.069 \\ -1.054 \\ -1.039 \end{pmatrix} \quad dv = \begin{pmatrix} -0.012 \\ 0 \\ 0.012 \end{pmatrix} \quad \beta v = \begin{pmatrix} -1.054 \\ 1.054 \\ 3.162 \end{pmatrix}$$

$$f(\eta, \eta\inf, b, c, d, \beta) := b\cdot\frac{\eta}{\eta\inf} + c\cdot\left(\frac{\eta}{\mu\inf}\right)^2 + d\cdot\left(\frac{\eta}{\eta\inf}\right)^3 + \\ + \beta\cdot\sqrt{\frac{\eta}{\eta\inf}} + \left(\frac{-b-c-d-\beta}{\ln(2)}\right)\cdot\ln\left(1+\frac{\eta}{\eta\inf}\right)$$

$$der1f(\eta,\eta\inf,b,c,d,\beta) := \frac{d}{d\eta}f(\eta,\eta\inf,b,c,d,\beta) \quad der2f(\eta,\eta\inf,b,c,d,\beta) := \frac{d^2}{d\eta^2}f(\eta,\eta\inf,b,c,d,\beta)^2$$

$$ecu(\eta,\eta\inf,b,c,d,\beta) := 2\cdot der2f(\eta,\eta\inf,b,c,d,\beta) + \eta\cdot der1f(\eta,\eta\inf,b,c,d,\beta) + \\ + 2\cdot f(\eta,\eta\inf,b,c,d,\beta)$$

```
rez := | n ← 10
       | nrc ← 1
       | for kη ∈ 1 .. nr
       |   | ηinf ← ηinfv_kη
       |   | for kβ ∈ 1 .. nr
       |   |   | β ← βv_kβ
       |   |   | for kd ∈ 1 .. nr
       |   |   |   | d ← dv_kd
       |   |   |   | for kc ∈ 1 .. nr
       |   |   |   |   | c ← cv_kc
       |   |   |   |   | for kb ∈ 1 .. nr
       |   |   |   |   |   | b ← bv_kb
       |   |   |   |   |   | ηinit ← 0
       |   |   |   |   |   | for i ∈ 1 .. n
       |   |   |   |   |   |   | ηtr_i ← ηinit + (ηinf − ηinit)/n · i
       |   |   |   |   |   |   | condecu ← (|ecu(ηtr_i, ηinf, b, c, d, β)|)^2
       |   |   |   |   |   |   | if condecu < 2.55 · 10^-5
       |   |   |   |   |   |   |   | int ← ∫_0^ηinf η · f(η, ηinf, b, c, d, β) dη
       |   |   |   |   |   |   |   | if |int − 1| < 0.01
       |   |   |   |   |   |   |   |   | ηinfi_nrc ← ηinf
       |   |   |   |   |   |   |   |   | bv_nrc ← b
       |   |   |   |   |   |   |   |   | cv_nrc ← c
       |   |   |   |   |   |   |   |   | dv_nrc ← d
       |   |   |   |   |   |   |   |   | βv_nrc ← β
       |   |   |   |   |   |   |   |   | inte_nrc ← int
       |   |   |   |   |   |   |   |   | ecua_nrc ← condecu
       |   |   |   |   |   |   |   |   | nrc ← nrc + 1
       | ( ηinfi  bv  cv  dv  βv  inte  ecua  nrc )
```

$$rez = \{\ \ \{104,1\}\ \ \{104,1\}\ \ \{104,1\}\ \ \{104,1\}\ \ \{104,1\}\ \ \{104,1\}\ \ \{104,1\}\ \ 105\}$$

$$\eta\inf i := rez_{1,1} \quad bv := rez_{1,2} \quad cv := rez_{1,3} \quad dv := rez_{1,4} \quad \beta v := rez_{1,5} \quad \lambda v := \frac{-bv - cv - dv - \beta v}{\ln(2)}$$

$$\operatorname{int} e := rez_{1,6} \quad nrc := rez_{1,8}$$

$$nrdata := nrc - 1 \quad nrdata = 104$$

$$cond := \operatorname{int} e - 1 \quad i := 1..nrdata \quad cond_i := \left|cond_i\right|$$

$$ecua := rez_{1,7}$$

$$re := augment(\eta\inf i, bv, cv, dv, \beta v, \lambda v, cond, ecua)$$

4. RESULTS AND INTERPRETATIONS

The values of constants that fulfill the conditions

$$\left|\int_0^{\eta_\infty} \varphi(\eta)\eta\, d\eta - 1\right| < 0.01 \tag{21}$$

$$\left|2\left(\varphi^2\right)'' + \eta\varphi' + 2\varphi\right|^2 < 2.55 \cdot 10^{-5} \tag{22}$$

have been obtained at the 57th computation line, as shown in Figure 1:

$\eta_{inf} = 2.515$ b = 0 c = -1.054 d = 0 β = 1.054 λ= 0

re =

	ηinfi	bv	cv	dv	βv	λv	cond	ecua
	1	2	3	4	5	6	7	8
47	2.515	-0.015	-1.054	0	1.054	0.022	$2.612\cdot10^{-3}$	$1.508\cdot10^{-5}$
48	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$2.347\cdot10^{-8}$
49	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$3.824\cdot10^{-8}$
50	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$4.245\cdot10^{-8}$
51	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$3.691\cdot10^{-8}$
52	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$2.456\cdot10^{-8}$
53	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$1.027\cdot10^{-8}$
54	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$7.222\cdot10^{-10}$
55	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$4.372\cdot10^{-9}$
56	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$3.141\cdot10^{-8}$
57	2.515	0	-1.054	0	1.054	0	$1.805\cdot10^{-5}$	$9.375\cdot10^{-8}$
58	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$7.134\cdot10^{-6}$
59	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$3.579\cdot10^{-6}$
60	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$1.111\cdot10^{-5}$
61	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$1.206\cdot10^{-5}$
62	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$8.37\cdot10^{-6}$
63	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$3.482\cdot10^{-6}$
64	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$2.922\cdot10^{-7}$
65	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$9.866\cdot10^{-7}$
66	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$7.169\cdot10^{-6}$
67	2.515	0.015	-1.054	0	1.054	-0.022	$2.576\cdot10^{-3}$	$2.001\cdot10^{-5}$

FIGURE 1. Results after computation

Running the presented program we obtained the series expansion of form

$$f(\eta) = -1.054 \cdot \left(\frac{\eta}{2.515}\right)^2 + 1.054 \cdot \sqrt{\frac{\eta}{2.515}} \tag{23}$$

which approximates the horizontal velocity of the boundary layer on a flat plate, given in part **a** of Figure 2. The exact solution is given in part **b**.

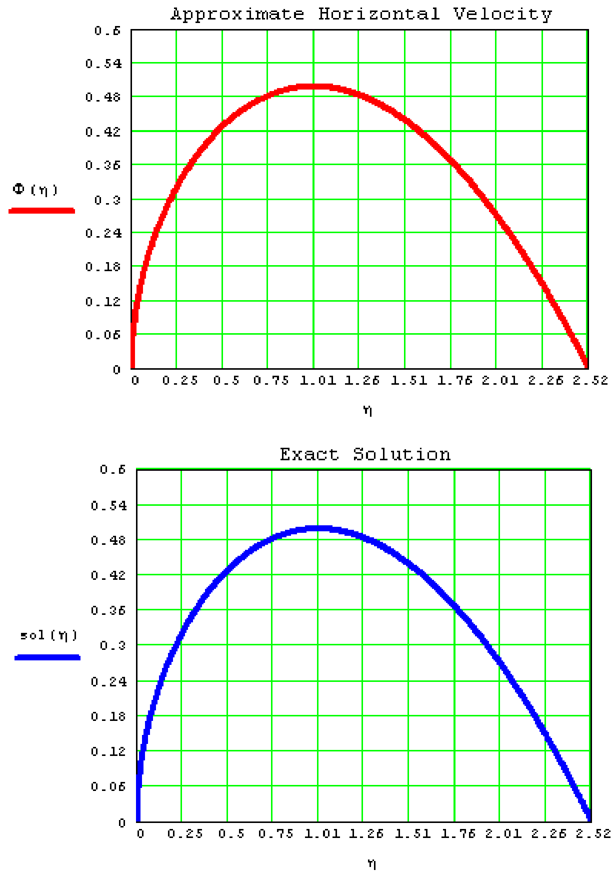

FIGURE 2. Comparison of approximate and exact values of the boundary layer parameters for the case of an incompressible flow along a flat plate

Analyzing this two functions, one can notice that the error committed by replacing the exact solution with its series expansion computed above is very small, the method being acceptable in practical cases.

5. CONCLUSIONS

The considerations presented in the paper may lead to the conclusion that the computational method obtained under the given conditions leads to the same solution like the exact one that describes the behavior of the boundary layer parameters for the case of an incompressible flow along a flat plate. This may be used in calculations of thermic parameters of motion that implies an ordinary differential equation and an integral condition.

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