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A COMPUTATIONAL STUDY UPON THE BURR 2-DIMENSIONAL DISTRIBUTION

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ABSTRACT: In this paper we generalyzed the one-dimensional Burr distribution at 2-dimensional space. We showed that this two-dimensional rule meets all conditions to be a continuous probability distribution, and it can be used in future modeling of certain statistical problems. Statistical inferences about the shape parameters have been discussed. Analysis of a real data set has been performed by giving the expresion of the probability density function applying the least squares method.
KEYWORDS: probability distribution , Burr distribution, least squares method, order statistics.

1. INTRODUCTION

Burr introduced twelve different forms of cumulative distribution functions for modeling data (see [2]). In probability theory and statistics, the Burr distribution is a probability distribution used in econometrics, which is concerned with the tasks of developing and applying quantitative or statistical methods in the study of economic principles (see [1, 3, 5, 7]). Studying several aspects of the one-parameter Burr-distribution [8, 9] one can observe that the Burr-distribution can be used quite effectively in modeling strength data and also modeling general lifetime data.

The aim of this paper is to generalyze the one-dimensional Burr distribution at 2-dimensional space. We present in the Section 2 the expresion of 2-D Burr distribution that we propose and the condition that this function has to meet in order to be a continuous probability distribution. Having a set of real input data, applying the least squares method we obtained the optimal parameters of the two-dimensional probability density function for the giving data set. The method is exposed in Section 3. Calculations have been done in MathCad calculation package. Section 4 includes the results and the conclusions of the paper.

2. DISTRIBUTIONAL PROPERTIES

The Burr distribution has the probability density function

$$B1(x,\alpha,\beta,\gamma)=\gamma\cdot\alpha\cdot\beta^{\alpha}\cdot x^{\gamma-1}\left(\beta+x^{\gamma}\right)^{-(\alpha+1)} \tag{1}$$

where $x \ge 0$ and $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ must be strictly positive shape parameters [4, 6].

The aspect of the probability density function of Burr distribution depends on the values of its parameters, as it can be seen in Figure 1. Its cumulative distribution function is depicted in Figure 2, for several positive shape parameters.

Our aim in this paper is to generalyze the one-dimensional Burr distribution at 2-dimensional space.

Let $B: R_+ \times R_+ \to R$ be the probability density function of two-dimensional Burr distribution, expressed by

$$B(x,y,\alpha1,\beta1,\gamma1,\alpha2,\beta2,\gamma2)=\frac{\gamma1\cdot\gamma2\cdot\alpha1\cdot\alpha2\cdot\beta1^{\alpha1}\cdot\beta2^{\alpha2}\cdot x^{\gamma1-1}\cdot y^{\gamma2-1}}{\left(\beta1+x^{\gamma1}\right)^{(\alpha1+1)}\cdot\left(\beta2+x^{\gamma2}\right)^{(\alpha2+1)}} \tag{2}$$

where $x \ge 0$, $y \ge 0$ and $\alpha1 > 0$, $\beta1 > 0$, $\gamma1 > 0$, $\alpha2 > 0$, $\beta2 > 0$, $\gamma2 > 0$.

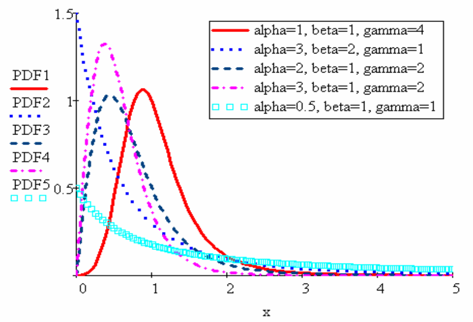

Figure 1. The Probability Density Function of the one dimensional Burr distribution for different shape parameters

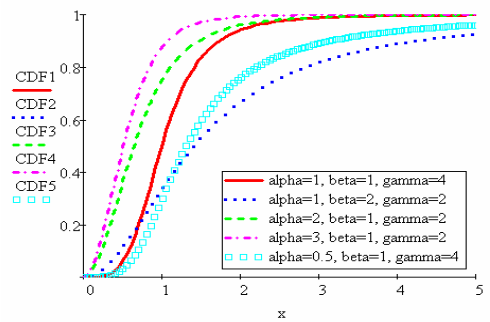

Figure 2. The Cumulative Distribution Function of the one dimensional Burr distribution for different shape parameters

In order to meet our purpose, the function (2) has to be a probability density [1, 6], and therefore has to meet the conditions

$$B(x,y,\alpha 1,\beta 1,\gamma 1,\alpha 2,\beta 2,\gamma 2)\geq 0 \tag{3}$$

and

$$\int_0^\infty\int_0^\infty B(x,y,\alpha 1,\beta 1,\gamma 1,\alpha 2,\beta 2,\gamma 2)dxdy=1. \tag{4}$$

Calculating the double integral for random strictly positive parameters, the result is

$$\int_0^{10^{307}}\int_0^{10^{307}} B(x,y,1.3,.9,1.1,1.2,1.3,1.3)\,dx\,dy = 1$$

The main goal of our study is to obtain the expression of the probability density for a 2-dimensional Burr distribution of a giving set of input values.

3. DATA ANALYSIS AND DISCUSSIONS

The set of numeric data that are going to be processed is given bellow, where the first two lines represent the values of the independent variables x and y, and the last line represents the independent variable u = f / 1000.

dat := augment(x, y, f) date := dat^T

date =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0.1	0.5	0.6	0.8	0.9	1.1	1.2	1.3	1.6	1.7	1.8	1.9	1.9	2.1	2.8
2	0.2	0.4	0.5	0.7	0.8	0.9	1.1	1.2	1.6	1.7	1.8	1.9	1.9	2.2	2.3
3	20	8.6	6.5	5.1	0.53	0.52	0.68	0.55	0.53	0.43	0.38	0.38	0.28	0.26	0.25

Constants $\alpha 1>0$, $\beta 1>0$, $\gamma 1>0$, $\alpha 2>0$, $\beta 2>0$, $\gamma 2>0$ are going to be determined using the least squares method, that assumes that the sum of the squares of the differences between the theoretical values of the function and the experimental values u, shall be minimum. The necessary condition is that the function derivatives with respect to $\alpha 1$, $\beta 1$, $\gamma 1$, $\alpha 2$, $\beta 2$, $\gamma 2$ to be null.

We shall find the parameters with the help of the program we are giving hereinafter:

ORIGIN ≡ 1

$$B(x,y,\alpha 1,\beta 1,\gamma 1,\alpha 2,\beta 2,\gamma 2) := \gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot x^{\gamma 1-1}\cdot y^{\gamma 2-1}\cdot\left(\beta 1+x^{\gamma 1}\right)^{-(\alpha 1+1)}\cdot\left(\beta 2+y^{\gamma 2}\right)^{-(\alpha 2+1)}$$

i := 1 .. length(x) n := length(x)

$\alpha 1 := .5$ $\beta 1 := .5$ $\gamma 1 := .3$

$\alpha 2 := .4$ $\beta 2 := .5$ $\gamma 2 := .2$

Given

$$\frac{d}{d\alpha 1}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$\frac{d}{d\beta 1}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$\frac{d}{d\gamma 1}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$\frac{d}{d\alpha 2}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$\frac{d}{d\beta 2}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$\frac{d}{d\gamma 2}\sum_{i=1}^{n}\left[\gamma 1\cdot\gamma 2\cdot\alpha 1\cdot\alpha 2\cdot\beta 1^{\alpha 1}\cdot\beta 2^{\alpha 2}\cdot(x_i)^{\gamma 1-1}\cdot(y_i)^{\gamma 2-1}\cdot\left[\beta 1+(x_i)^{\gamma 1}\right]^{-(\alpha 1+1)}\cdot\left[\beta 2+(y_i)^{\gamma 2}\right]^{-(\alpha 2+1)}-u_i\right]^2=0$$

$$sol := \mathrm{Find}(\alpha 1,\beta 1,\gamma 1,\alpha 2,\beta 2,\gamma 2) \qquad sol = \begin{pmatrix}100.6\\113.138\\0.05\\53.096\\67.961\\0.069\end{pmatrix}$$

$$\alpha 1e := sol_1 \quad \beta 1e := sol_2 \quad \gamma 1e := sol_3$$
$$\alpha 2e := sol_4 \quad \beta 2e := sol_5 \quad \gamma 2e := sol_6$$

4. RESULTS AND INTERPRETATIONS

The program code presented hereinbefore evaluates the first derivatives of the probability density (2) with respect to parameters $\alpha 1$, $\beta 1$, $\gamma 1$, $\alpha 2$, $\beta 2$ and $\gamma 2$.

We have started the procedure of calculating derivatives with six initial guess values, far away from the solution and the program compute the next values for the Burr probability density function in two-dimensional space:

$$\alpha 1e = 100.6\ ,\quad \beta 1e = 113.138\ ,\quad \gamma 1e = 0.05$$
$$\alpha 2e = 53.096\ ,\quad \beta 2e = 67.961\ ,\quad \gamma 2e = 0.069\ .$$

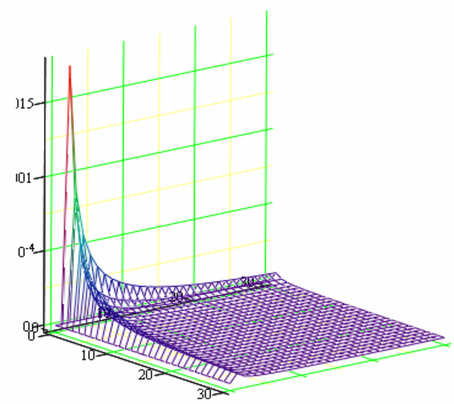

Figure 3. The Probability Density Function of the two-dimensional Burr distribution

These values of $\alpha 1e$, $\beta 1e$, $\gamma 1e$, $\alpha 2e$, $\beta 2e$, $\gamma 2e$ substituted in expression (2) lead to the two-dimensional Burr probability density function

$$Burr(x,y)=\frac{18.428\cdot 113.138^{100.6}\cdot 67.961^{53.096}\cdot x^{-0.95}\cdot y^{-0.931}}{\left(113.138+x^{0.05}\right)^{101.6}\cdot\left(67.961+x^{0.069}\right)^{54.096}} \tag{5}$$

The graphic representation of the two-dimensional probability density function which has been obtained for the given input data set is depicted in Figure 3, with its contours showed in Figure 4.

The correlation coefficient and the standard deviation of the two-dimensional probability function (5) are

$$r := \sqrt{1 - \frac{\sum_i \left(u_i - B\left(x_i, y_i, \alpha 1e, \beta 1e, \gamma 1e, \alpha 2e, \beta 2e, \gamma 2e\right)\right)^2}{\sum_i \left(u_i - \text{mean}(u)\right)^2}} \qquad r = 0.887 ,$$

where $\text{mean}(u)$ represents the mean value of the independent variable u, and respectively

$$sd := \sqrt{\frac{1}{n} \cdot \left[\sum_i \left(u_i - B\left(x_i, y_i, \alpha 1e, \beta 1e, \gamma 1e, \alpha 2e, \beta 2e, \gamma 2e\right)\right)^2\right]} \qquad sd = 2.412 \times 10^{-3} .$$

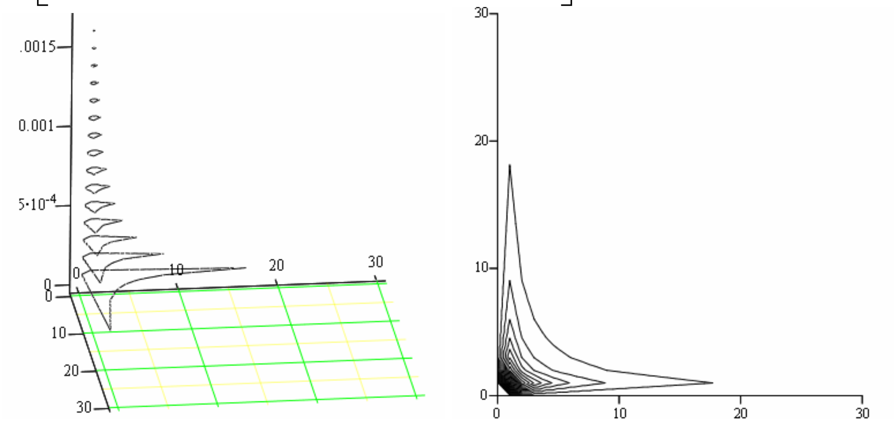

Figure 4. The contours of the probability density function of the two-dimensional Burr distribution:
a – 3D view, b – 2D view

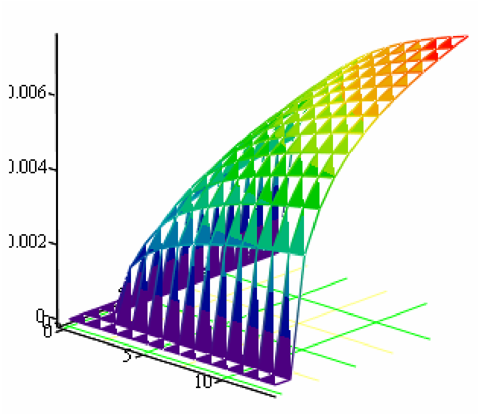

Figure 5. The cumulative distribution function of the two-dimensional Burr distribution

The graph of the cumulative distribution function that describes the two-dimensional statistical distribution is given in Figure 5.

The 2-dimensional Burr distribution is a continuous probability distribution and it can be used in modeling certain problems.

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