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ALGORITHMIZATION OF INTERVAL STRUCTURAL ANALYSIS

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ABSTRACT:
The paper presents a non-traditional handling of uncertainties in material, geometric and load parameters in linear structural analysis, mainly in static and modal-spectral analysis. Uncertainties are introduced as bounded possible values – intervals. The main goal has been to propose algorithms for interval computations on FEM models suggested by authors. An application of the chosen approaches is going to be presented; the first one is a simple combination of only inf-values or only sup-values; the second one presents the full combination of all inf-sup values; the third one uses the optimization process as a tool for finding out a inf-sup solution and the last one is Monte Carlo technique as a comparison tool.
KEYWORDS:
interval arithmetic, uncertain parameters, MATLAB, Monte Carlo, optimization

1. INTRODUCTION

In the last decade, there has been an increased interest in the modeling and analysis of engineering systems under uncertainties. To obtain reliable results for the solutions of engineering problems, exact values for the parameters of the model equations should be available. In the reality, however, those values often can not be provided, and the models usually show a rather high degree of uncertainty. Computational mechanics, for example, encounters uncertainties in geometric, material and load parameters as well as in the model itself and in the analysis procedure too. For that reason, the responses, such as displacements, stresses, resonant frequency, or other dynamic characteristics, will usually show some degree of uncertainty [10,11,12]. It means that the obtained result using one specific value as the most significant value for an uncertain parameter cannot be considered as representative for the whole spectrum of possible results.

It is generally known that probabilistic modeling and statistical analysis are well established for modeling of mechanical systems with uncertainties. In addition, a number of non-probabilistic computational techniques have been proposed, e.g. fuzzy set theory [1,9,10,11,13], interval approach [3,4,5,8,16,17], imprecise probabilities [2,9,15,17] etc. The growing interest in these approaches originated from a criticism of the credibility of probabilistic approach when input dates are insufficient [Zhang]. It is argued that the new non-probabilistic treatments could be more appropriate in the modeling of the vagueness.

The uncertainty is considered as unknown but bounded with lower and upper bounds. The interval numbers derived from the experimental data or expert knowledge can then take into account the uncertainties in the model parameters, model inputs etc. By this technique, the complete information about the uncertainties in the model may be included and one can demonstrate how these uncertainties are processed by the calculation procedure in MATLAB.

2. FUNDAMENTAL PRINCIPLES OF INTERVAL ARITHMETIC

Interval arithmetic was developed by R. E. Moore [4] while studying the propagation and control of truncation and rounding off the error, using floating point arithmetic [5] on a digital computer. Moore was able to generalize this work into the arithmetic independence of machine considerations [17].

In this approach, an uncertain number is represented by an interval of real numbers. An interval number [5,6,7] is a closed set **R** that includes the possible range of an unknown real number where **R** denotes the set of real numbers. Therefore, a real interval is a set of the form

$$\mathbf{x} = [\underline{x}, \overline{x}] = \{x \in \mathbf{R} : \underline{x} \le x \le \overline{x}\}, \tag{1}$$

where $\underline{x}$ and $\overline{x}$ are the lower (*infimum*) and upper (*supremum*) bounds of the interval number $\mathbf{x}$ respectively, and the bounds are elements of **R** with $\underline{x} \le \overline{x}$. Definition of real intervals and operations with intervals could be found in a number of references [5, 6]. Let's define basic properties of interval number that have been inbuilt in INTLAB [7]:

- the *midpoint* of $\mathbf{x}$: $\operatorname{mid}(\mathbf{x}) = \breve{x} = \frac{1}{2}(\underline{x} + \overline{x})$,
- the *radius* of $\mathbf{x}$: $\operatorname{rad}(\mathbf{x}) = \frac{1}{2}(\overline{x} - \underline{x})$, (2)
- the *absolute value* or the *magnitude* of $\mathbf{x}$: $|\mathbf{x}| = \operatorname{mag}(\mathbf{x}) = \max\{|\tilde{x}| : \tilde{x} \in \mathbf{x}\}$,
- the *magnitude* of $\mathbf{x}$: $\operatorname{mig}(\mathbf{x}) = \min\{|\tilde{x}| : \tilde{x} \in \mathbf{x}\}$.

Given $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$, the four elementary operations are defined by

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}],\\
\mathbf{x} - \mathbf{y} &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}],\\
\mathbf{x} \times \mathbf{y} &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}],\\
\mathbf{x} \div \mathbf{y} &= \mathbf{x} \times 1/\mathbf{y},\\
1/\mathbf{x} &= [1/\overline{x}, 1/\underline{x}] \quad \text{if } \underline{x} > 0 \text{ or } \overline{x} < 0.
\end{aligned} \tag{3}$$

For the elementary interval operations, division by an interval containing zero is not defined. It is often useful to remove this restriction to give so called *extended interval arithmetic* [5,6,7]. Extended interval arithmetic leads to the following rules. If $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ with $\underline{y} \le 0 \le \overline{y}$ and $\underline{y} < \overline{y}$, then the rules for division are as follows

$$\mathbf{x}/\mathbf{y} = \begin{cases}
[\overline{x}/\underline{y}, \infty] & \text{if } \overline{x} \le 0 \text{ and } \overline{y} = 0\\
[-\infty, \overline{x}/\overline{y}] \cup [\overline{x}/\underline{y}, \infty] & \text{if } \overline{x} \le 0 \text{ and } \underline{y} < 0 < \overline{y}\\
[-\infty, \overline{x}/\overline{y}] & \text{if } \overline{x} \le 0 \text{ and } \underline{y} = 0\\
[-\infty, \infty] & \text{if } \underline{x} < 0\overline{x}\\
[-\infty, \underline{x}/\underline{y}] & \text{if } \underline{x} \ge 0 \text{ and } \overline{y} = 0\\
[-\infty, \underline{x}/\underline{y}] \cup [\underline{x}/\overline{y}, \infty] & \text{if } \underline{x} \ge 0 \text{ and } \underline{y} < 0 < \overline{y}\\
[\underline{x}/\overline{y}, \infty] & \text{if } \underline{x} \ge 0 \text{ and } \underline{y} = 0
\end{cases}. \tag{4}$$

For further rules for extended interval arithmetic, see [5,6].

Interval Matrix Analysis and Comparison with Monte Carlo

Considering uncertain parameters in interval form, we'll realize comparison study of the few basic matrix operations:

- solution of the linear equations system (INTLAB function - *verifylnss*) [7],
- solution of the eigenvalue problem (INTLAB function – *verifyeig*) [7].

The alternative avenue of the interval arithmetic is to use the Monte Carlo technique. With the advent of recent computational facilities, this method becomes attractive. The results are determined from the series of numerical analyses (approximately 1000–10000 iterations). It is recommended to generate the random values with the uniform distribution.

Example 1

Let us consider uncertain linear algebraic system

$$S \cdot y = b \tag{5}$$

or

$$\begin{bmatrix} \langle 7{,}60 \quad 8{,}40 \rangle & \langle 4{,}75 \quad 5{,}25 \rangle & \langle 1{,}90 \quad 2{,}10 \rangle \\ \langle 2{,}85 \quad 3{,}15 \rangle & \langle 8{,}55 \quad 9{,}45 \rangle & \langle 0{,}95 \quad 1{,}05 \rangle \\ \langle 4{,}75 \quad 5{,}25 \rangle & 0 & \langle 10{,}45 \quad 11{,}55 \rangle \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \langle 13{,}5 \quad 16{,}5 \rangle \\ 0 \\ \langle 19{,}8 \quad 24{,}2 \rangle \end{bmatrix}.$$

Thereafter, the interval solution by "*verifylnss*" procedure versus Monte Carlo simulation is presented in Table 1. The comparison has been realized with the usage of midpoint residual vector $\mathbf{r}_{Midpoint}$ and radius residual vector $\mathbf{r}_{Residual}$ expressed in %, e.g.

$$r_{Midpoint} = \left| \frac{mid(y_{Intlab}) - mid(y_{MC})}{mid(y_{Intlab})} \right| \cdot 100 \ \% \tag{6}$$

and

$$r_{Radius} = \left| \frac{rad(y_{Intlab}) - rad(y_{MC})}{rad(y_{Intlab})} \right| \cdot 100 \ \% . \tag{7}$$

Table 1.

y_{Intalb} - solution by interval arithmetic *(verifylnss)*	y_{MC} - solution by Monte Carlo simulation	$r_{Midpoint}$ - midpoint residual vector in %	r_{Radius} - radius residual vector in %
$\begin{bmatrix} \langle 1{,}3887 \quad 2{,}8782 \rangle \\ \langle -1{,}1409 \quad -0{,}5103 \rangle \\ \langle 0{,}3370 \quad 1{,}7235 \rangle \end{bmatrix}$	$\begin{bmatrix} \langle 1{,}6799 \quad 2{,}6295 \rangle \\ \langle -1{,}0479 \quad -0{,}6469 \rangle \\ \langle 0{,}6042 \quad 1{,}4579 \rangle \end{bmatrix}$	$\begin{bmatrix} 0{,}99 \\ 8{,}13 \\ 5{,}95 \end{bmatrix}$	$\begin{bmatrix} 36{,}2 \\ 8{,}2 \\ 29{,}7 \end{bmatrix}$

It should be noted that the difference in radiuses is significant for this fundamental mathematic problem.

Example 2

Let's solve now the "eigenproblem"

$$(S - \lambda_i \cdot I) \cdot v_i = 0 \tag{8}$$

or

$$\left(\begin{bmatrix} \langle 7{,}60 \quad 8{,}40 \rangle & \langle 4{,}75 \quad 5{,}25 \rangle & \langle 1{,}90 \quad 2{,}10 \rangle \\ \langle 2{,}85 \quad 3{,}15 \rangle & \langle 8{,}55 \quad 9{,}45 \rangle & \langle 0{,}95 \quad 1{,}05 \rangle \\ \langle 4{,}75 \quad 5{,}25 \rangle & 0 & \langle 10{,}45 \quad 11{,}55 \rangle \end{bmatrix} - \lambda_i \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Assuming "*verifyeig*" solution, we obtain following results, spectral matrix

$$\lambda = \begin{bmatrix} \langle 3{,}2431 \quad 5{,}1197 \rangle & 0 & 0 \\ 0 & \langle 8{,}2120 \quad 10{,}1563 \rangle & 0 \\ 0 & 0 & \langle 13{,}3885 \quad 15{,}8804 \rangle \end{bmatrix}$$

and modal matrix

$$V = \begin{bmatrix} \langle -0{,}754 & -0{,}755 \rangle & \langle -0{,}4668 & -0{,}1542 \rangle & \langle -0{,}7436 & -0{,}3261 \rangle \\ \langle 0{,}2232 & 0{,}4861 \rangle & \langle -0{,}5786 & -0{,}2524 \rangle & \langle -0{,}5984 & -0{,}2323 \rangle \\ \langle 0{,}3790 & 0{,}7268 \rangle & \langle 0{,}8550 & 0{,}856 \rangle & \langle -0{,}7358 & -0{,}7358 \rangle \end{bmatrix}.$$

Let's compare interval results with Monte Carlo simulation. Thereafter, by this approach, we obtain spectral matrix

$$\lambda_{MC} = \begin{bmatrix} \langle 3{,}5963 \quad 4{,}6910 \rangle & 0 & 0 \\ 0 & \langle 8{,}6447 \quad 9{,}7308 \rangle & 0 \\ 0 & 0 & \langle 14{,}0767 \quad 15{,}1879 \rangle \end{bmatrix}$$

and modal matrix

$$V = \begin{bmatrix} \langle -0{,}7501 & -0{,}7535 \rangle & \langle -0{,}3176 & -0{,}3056 \rangle & \langle -0{,}5312 & -0{,}5298 \rangle \\ \langle 0{,}3459 & 0{,}3640 \rangle & \langle -0{,}4176 & -0{,}4030 \rangle & \langle -0{,}4170 & -0{,}3981 \rangle \\ \langle 0{,}5521 & 0{,}5591 \rangle & \langle 0{,}8557 & 0{,}8583 \rangle & \langle -0{,}7479 & -0{,}7385 \rangle \end{bmatrix}.$$

The comparison of the spectral matrices has been again realized through the use of midpoint residual vector and radius residual vector expressed in % (see Table 2). The differences in radiuses are also significant and acknowledge the overestimating effect of interval arithmetic.

Table 2.

Eigenvalue	Midpoint residual values in %	Radius residual values in %
λ_1	0.9028	41.6658
λ_2	-0.0392	44.1393
λ_3	0.0147	55.4075

3. PROPOSITION AND APPLICATION OF NUMERICAL METHODS

During the solving of the particular tasks in the engineering practice using the interval arithmetic application on the solution of numerical mathematics and mechanical problems, the problem known as the overestimate effect is encountered. Its elimination is possible only in the case of meeting the specific assumptions, mainly related to the time efficiency of the computing procedures. Now, we will try to analyze some solution approaches already used or proposed by the authors. We will consider the following methods:

- Monte Carlo method (MC) [15, 17],
- method of a solution evaluation in marginal values of interval parameters – infimum and supremum (COM1) [10],
- method of a solution evaluation for all marginal values of interval parameters – all combinations of infimum and supremum (COM2),
- method of infimum and supremum searching using some optimizing technique application (OPT) [9],
- direct application of the interval arithmetic using INTLAB – MATLAB's toolbox (INTL), [7].

Monte Carlo method (MC) is a time consuming but reliable solution tool. Various combinations of the uncertain parameter deterministic values are generated (Figure 1) and after the subsequent solution in the deterministic sense we obtain a complete set of results processed in an appropriate manner. Infimum and supremum calculation is following

$$\inf(F) = \text{min of all results } F(p_i),\ \text{ where } i = 1,\dots m \ \text{ and } \ m \approx 5000 \div 100000\,,$$

and

$$\sup(F) = \text{max of all results } F(p_i),\ \text{ where } i = 1,\dots m \ \text{ and } \ m \approx 5000 \div 100000\,.$$

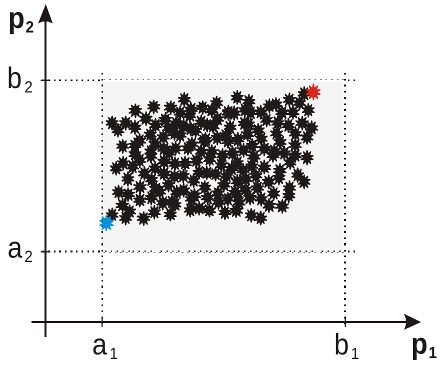

Figure 1. MC generation of the realizations map for two interval parameters

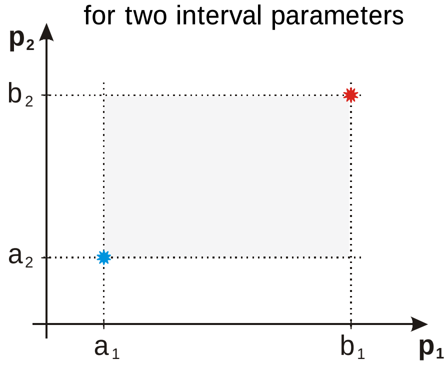

Figure 2. COM1 - realizations map for 2 interval parameters

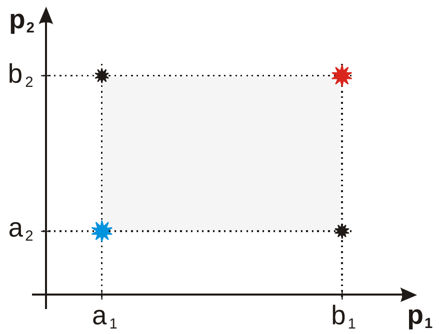

Figure 3. COM2 - realizations map for two interval parameters

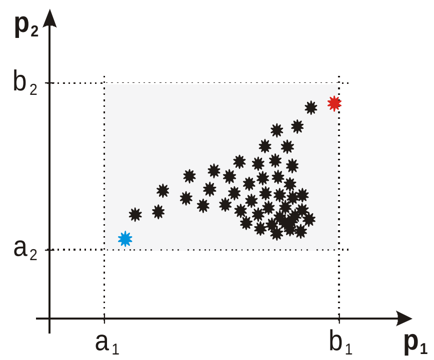

Figure 4. OPT- realizations map for two interval parameters

Second method application (COM1), i.e. solution evaluation in marginal values of interval parameters has its physical meaning for many engineering problems. We consider an approach where the extreme output values are obtained by the application of the extreme parameter values on input. That means that the final solution infimum is obtained using the deterministic analysis for infimum of uncertain input parameters and in the opposite way, the final supremum will be obtained using the deterministic analysis for supremum of the input uncertain parameters (see Figure 2). Inf-sup calculation is

$\inf(F) = \min \text{ of } [F(\underline{p}), F(\overline{p})]$ and

$\sup(F) = \max \text{ of } [F(\underline{p}), F(\overline{p})]$. (9)

The third approach COM2 which is also based on the set of the deterministic analyses appears as the more suitable one. The marginal interval parameter values are considered again but the infimums and supremums are also combined (Figure 3). The method provides satisfying results and can be marked as reliable, even if there is still a doubt about the existence of the extreme solution for the uncertain parameter inner values. Solution for two interval numbers $p_1 = \langle a_1 \;\; b_1 \rangle$ and $p_2 = \langle a_2 \;\; b_2 \rangle$ may be found by this computational way

$$\inf(F) = \min \text{of} [F(a_1\ a_2),\ F(a_1\ b_2),\ F(b_1\ a_2),\ F(b_1\ b_2).] \qquad (10)$$

$$\sup(F) = \max \text{of} [F(a_1\ a_2),\ F(a_1\ b_2),\ F(b_1\ a_2),\ F(b_1\ b_2).] \qquad (11)$$

The method of the infimum and supremum solution searching using the optimization techniques (OPT, Figure 4) is proposed by the authors as an alternative to the first and to the third method. It should eliminate a big amount of analyses in the first method and also eliminates the problem with the possibility of the infimum and supremum existence inside of the interval parameters for the deterministic values. Computational process for two interval numbers $p_1 = \langle a_1 \;\; b_1 \rangle$ and $p_2 = \langle a_2 \;\; b_2 \rangle$ may be found as follows

$$\inf(F) = F(p_{OPT}) \text{ i.e. find } p_{OPT} \text{ so that } F(p_{OPT}) \rightarrow \min. \qquad (12)$$

$$\sup(F) = F(p_{OPT}) \text{ i.e. find } p_{OPT} \text{ so that } F(p_{OPT}) \rightarrow \max. \qquad (13)$$

The authors used also the interval arithmetic principles implemented in INTLAB as another computing tool. However, the overestimate effect mentioned above for the significant uncertainties causes considerable problems and the possibilities of INTLAB using are therefore very restricted. INTLAB using makes sense particularly for simple problems because of the results obtaining speed.

Example 3

Let's apply the presented numerical methods into the solution of the following system

$$\begin{bmatrix} <0,1> & 1 \\ -2 & <-1,2> \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} <0,2> \\ 0 \end{bmatrix}.$$

The solution is shown on Figure 5 and the comparison of the used methods is summarized in Table 3.

Table 3. Results obtained by the proposed numerical methods

	exact solution	1st method MC	2nd method COM1	3rd method COM2	4th method OPT	5th method INTLAB
Y_1	<-2 2>	<-1,95 1,94>	<-1 1>	<-2 2>	<-2 0>	<-4,78 4,78>
Y_2	<0 4>	<0 3,91>	<0 2>	<0 4>	<0 4>	<-5,1 5,1>

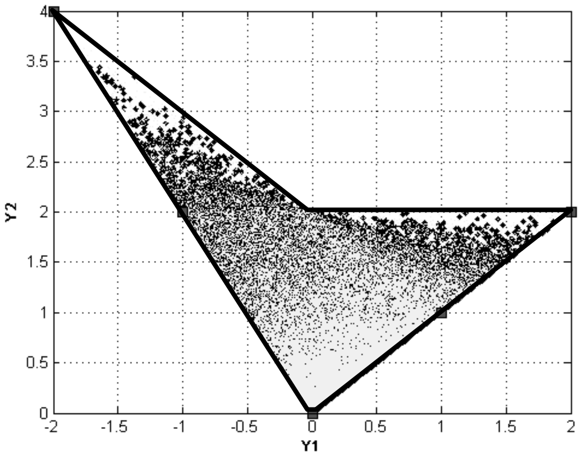

Figure 5. Map of the interval linear equations system solution

In general, solving of the interval equations system may be a complicated problem (mainly for large dimension, [5]). Good information about the solution set is obtained using Monte Carlo method. The other methods determine only the marginal values with bigger or smaller inaccuracies. The 1st and the 3rd used methods are particularly suitable for the equations systems solving according to the experiences of the authors. In case of the fourth method application (the optimization method) there is a problem with a formulation of the appropriate test function which would properly describe searching of multiple inf or sup solutions. INTLAB usage appears to be unsuitable because the parameter uncertainty is rather strong.

Example 4

Let's compare the proposed interval computational methods during solving of the following eigenvalues problem

a) with a small signification of the parameters uncertainty, e.g.

$$\left(\begin{bmatrix} \langle 19400\ 19500 \rangle & -\langle 9400\ 9450 \rangle \\ -\langle 9400\ 9450 \rangle & \langle 9400\ 9450 \rangle \end{bmatrix} - \lambda_i \cdot \begin{bmatrix} \langle 20\ 20{,}1 \rangle & 0 \\ 0 & \langle 18\ 18{,}1 \rangle \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) and with a larger signification of the parameters uncertainty, e.g.

$$\left(\begin{bmatrix} \langle 19400\ 21400 \rangle & -\langle 9400\ 10400 \rangle \\ -\langle 9400\ 10400 \rangle & \langle 9400\ 10400 \rangle \end{bmatrix} - \lambda_i \cdot \begin{bmatrix} \langle 20\ 25 \rangle & 0 \\ 0 & \langle 18\ 22 \rangle \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Inf-sup results has been compiled into Table 4. Graphic representation of the results is shown on Figure 6.

Table 4. Results obtained by the proposed numerical methods

		1st method	2nd method	3rd method	4th method	5th method
A	λ_1	<201 203>	<202,3 202,5>	<201 203>	<201 203>	<199 206>
	λ_2	<1284 1296>	<1289 1290>	<1283 1296>	<1283 1296>	<1268 1312>
B	λ_1	<167 220>	<181 202>	<164 223>	<164 223>	<95 287>
	λ_2	<1059 1415>	<1147 1290>	<1039 1425>	<1038 1425>	<600 1827>

Solving of the eigenvalues problem as a frequent task of the solid mechanics has demonstrated the facilities of the particular approaches. Monte Carlo method gives usually good information about the solution set. The other methods determine only the marginal values. For the eigenvalues analysis of the systems with the interval parameters the authors recommend according to their experiences to use particularly the 1st, 3rd and 4th method. All of them give satisfying results but especially the 3rd and 4th ones appear to be the most suitable.

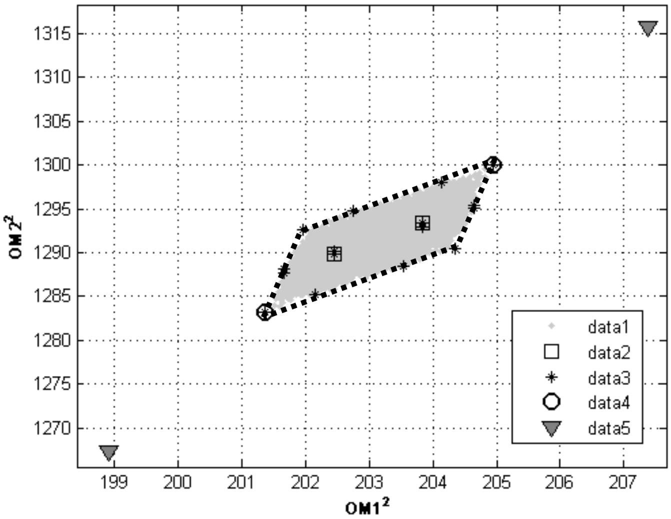

Figure 6. Solution of the set eigenvalues with the small uncertainties
data1 – Monte Carlo analysis
data2 – COM1, evaluation in marginal values (only inf or only sup)
data3 – COM2, evaluation for all marginal values–all combinations
data4 – OPT, inf and sup searching using optimizing technique application
data5 – direct application of the interval arithmetic using INTLAB
------ – boundary of the all possible solutions set

From the efficiency perspective, the third method can be considered as the best even if there exists a risk of loosing the inner interval number solutions. Usage of the second and the fifth method (INTLAB) is determined by the uncertainty importance (the interval size). As it is presented, all methods are acceptable in the case of a minor uncertainty (solution A).

4. INTERVAL FINITE ELEMENTS ANALYSIS (IFEA)

The finite element method (FEM) [1, 2, 8, 10,15] is a very popular tool for a complicated structural analysis. The ability to predict the behavior of a structure under static or dynamic loads is not only of a great scientific value, it is also very useful from an economical point of view. A reliable FE analysis could reduce the need for prototype production and therefore significantly reduce the associated design validation cost.

It is sometimes very difficult to define a reliable FE model for realistic mechanical structures when a number of its physical properties is uncertain. Particularly, in the case of FE analysis, the mechanical properties of the used materials are very hard to predict, and therefore an important source of uncertainty. Reliable validation can only be based on an analysis which takes into account all uncertainties that could cause this variability. It is the aim of this part to incorporate the most important uncertainties in FE analysis.

According to the character of the uncertainty, we can define a structural uncertainty (geometrical and material parameters) and uncertainty in load (external forces, etc.). The structural uncertainty parameters are usually written into matrix $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and the interval static (time independent) FE analysis may be formulated as follows

$$\mathbf{K}(\mathbf{x}) \cdot \mathbf{u} = \mathbf{f}(\mathbf{x}) \quad \text{or} \quad [\underline{\mathbf{K}}, \overline{\mathbf{K}}] \cdot [\underline{\mathbf{u}}, \overline{\mathbf{u}}] = [\underline{\mathbf{f}}, \overline{\mathbf{f}}] \tag{14}$$

where $\underline{\mathbf{K}}, \overline{\mathbf{K}}$ are the infimum and supremum matrices of the stiffness matrix **K**, $\underline{\mathbf{u}}, \overline{\mathbf{u}}$ are the infimum and supremum vectors of the displacements vector **u**, $[\underline{\mathbf{f}}, \overline{\mathbf{f}}]$ are the infimum and supremum vectors of the loading vector **f**. Considering a dynamic conservative system, it is possible to obtain the interval modal and spectral matrices using the solution of

$$[\mathbf{K}(\mathbf{x}) - \lambda_j \cdot \mathbf{M}(\mathbf{x})] \cdot \mathbf{v}_j = \mathbf{0} \quad \text{or} \quad ([\underline{\mathbf{K}}, \overline{\mathbf{K}}] - [\underline{\lambda}_j, \overline{\lambda}_j] \cdot [\underline{\mathbf{M}}, \overline{\mathbf{M}}]) \cdot [\underline{\mathbf{v}}_j, \overline{\mathbf{v}}_j] = \mathbf{0}, \tag{15}$$

where $\underline{\lambda}_j, \overline{\lambda}_j$ and $\underline{\mathbf{v}}_j, \overline{\mathbf{v}}_j$ are the *j*-th eigenvalue with corresponding eigenvector, $\underline{\mathbf{K}}, \overline{\mathbf{K}}, \underline{\mathbf{M}}, \overline{\mathbf{M}}$ are of course the infimum and supremum of the mass and stiffness matrices. The application of the classic interval arithmetic for FE analysis is very limited. Its "overestimation" grows with the problem size (the dimension of the system matrices) and has not a physical foundation in the reality. Therefore, it is efficient to apply the previous numerical methods.

Application of the **Monte Carlo** method in IFEA may be realized as follows:

Static analysis

1. step: generation of the random matrix (uniform distribution)

$$\mathbf{X}_{MC} = [\mathbf{x}_1, \ldots .\mathbf{x}_m], \qquad (m \approx 5000 \div 100000),$$

2. step: solution of

$$\mathbf{U}_{MC} = [\mathbf{K}(\mathbf{x}_1)^{-1} \cdot \mathbf{f}(\mathbf{x}_1), \ldots, \mathbf{K}(\mathbf{x}_m)^{-1} \cdot \mathbf{f}(\mathbf{x}_m)],$$

3. step:

- infimum calculation $\qquad \underline{u}_i = \inf(i^{th} \text{ row of } \mathbf{U}_{MC}) \to \underline{\mathbf{u}},$
- supremum calculation $\qquad \overline{u}_i = \sup(i^{th} \text{ row of } \mathbf{U}_{MC}) \to \overline{\mathbf{u}}.$

Eigenvalues analysis

1. step: generation of the random matrix (uniform distribution)

$$\mathbf{X}_{MC} = [\mathbf{x}_1, \ldots .\mathbf{x}_m], \qquad (m \approx 5000 \div 100000),$$

2. step: solution of

$$\boldsymbol{\lambda}_{j\_MC} \to [\mathbf{K}(\mathbf{x}_j) - \boldsymbol{\lambda}_{j\_MC} \cdot \mathbf{M}(\mathbf{x}_j)] \cdot \mathbf{V}_j = \mathbf{0} \quad \text{for } j = 1 \ldots m,$$

3. step:

- infimum calculation of the *i-th* eigenvalue $\qquad \underline{\lambda}_i = \inf(i^{th} \text{ row of } \boldsymbol{\lambda}_{MC}),$
- supremum calculation of the *i-th* eigenvalue $\overline{\lambda}_i = \sup(i^{th} \text{ row of } \boldsymbol{\lambda}_{MC}).$

In the case of **COM1**, the numerical approach implementation to IFEA is following:

Static analysis

- infimum calculation $\qquad \underline{\mathbf{u}} = \mathbf{K}(\underline{\mathbf{x}})^{-1} \cdot \mathbf{f}(\underline{\mathbf{x}}),$
- supremum calculation $\qquad \overline{\mathbf{u}} = \mathbf{K}(\overline{\mathbf{x}})^{-1} \cdot \mathbf{f}(\overline{\mathbf{x}}).$

Eigenvalues analysis

- infimum calculation $\qquad \underline{\boldsymbol{\lambda}} \to [\mathbf{K}(\underline{\mathbf{x}}) - \underline{\boldsymbol{\lambda}} \cdot \mathbf{M}(\underline{\mathbf{x}})] \cdot \underline{\mathbf{V}} = \mathbf{0},$
- supremum calculation $\qquad \overline{\boldsymbol{\lambda}} \to [\mathbf{K}(\overline{\mathbf{x}}) - \overline{\boldsymbol{\lambda}} \cdot \mathbf{M}(\overline{\mathbf{x}})] \cdot \overline{\mathbf{V}} = \mathbf{0}.$

COM1 doesn't give the correct results every time. We can obtain more proper results using **COM2**. Its computational process for IFEA is:

Static analysis

1. step: calculation of realizations matrix $\mathbf{X}_2$, i.e. 2^n inf-sup combinations, (n – number of uncertain system parameters),

$$\mathbf{X}_{COM2} = [\mathbf{x}_1, \ldots .\mathbf{x}_m], \qquad (m = 2^n),$$

2. step: solution of

$$\mathbf{U}_{COM2} = [\mathbf{K}(\mathbf{x}_1)^{-1} \cdot \mathbf{f}(\mathbf{x}_1), \ldots, \mathbf{K}(\mathbf{x}_m)^{-1} \cdot \mathbf{f}(\mathbf{x}_m)],$$

3. step:

- infimum calculation $\qquad \underline{u}_i = \inf(i^{th} \text{ row of } \mathbf{U}_{MC}) \to \underline{\mathbf{u}},$
- supremum calculation $\qquad \overline{u}_i = \sup(i^{th} \text{ row of } \mathbf{U}_{MC}) \to \overline{\mathbf{u}}.$

Eigenvalues analysis

1. step: calculation of realizations matrix $\mathbf{X}_2$, i.e. 2^n inf-sup combinations, (n – number of uncertain system parameters),

$$\mathbf{X}_{COM2} = [\mathbf{x}_1, \ldots .\mathbf{x}_m], \qquad (m = 2^n),$$

2. step: solution of

$$\boldsymbol{\lambda}_{j\_COM2} \to [\mathbf{K}(\mathbf{x}_j) - \boldsymbol{\lambda}_{j\_COM2} \cdot \mathbf{M}(\mathbf{x}_j)] \cdot \mathbf{V}_j = \mathbf{0} \quad \text{for } j = 1 \ldots m,$$

3. step:

- infimum calculation of the *i-th* eigenvalue $\quad \underline{\lambda}_i = \inf(i^{th} \text{ row of } \boldsymbol{\lambda}_{COM2}),$
- supremum calculation of the *i-th* eigenvalue $\quad \overline{\lambda}_i = \sup(i^{th} \text{ row of } \boldsymbol{\lambda}_{COM2}).$

Generally, the infimum or supremum are not found only in the boundary points (COM1, COM2) but also in the inner domain of the solution set (OPT). To find the inf-sup solution using the approach **OPT** means to solve the optimizing problem described as follows:

Static analysis
- infimum calculation

$$\underline{u}_i(\mathbf{x}_{OPT}) \rightarrow \text{minimize value of } i^{th} \text{ member of } [\mathbf{K}(\mathbf{x})^{-1} \cdot \mathbf{f}(\mathbf{x})]),$$

- supremum calculation

$$\overline{u}_i(\mathbf{x}_{OPT}) \rightarrow \text{maximize value of } i^{th} \text{ member of } [\mathbf{K}(\mathbf{x})^{-1} \cdot \mathbf{f}(\mathbf{x})]).$$

Eigenvalues analysis
- infimum calculation of the *i*-th eigenvalue

$$\underline{\lambda}_i(\mathbf{x}_{OPT}) \rightarrow \text{minimize value of } \lambda_i \text{ for eq.: } [\mathbf{K}(\mathbf{x}) - \boldsymbol{\lambda}_i \cdot \mathbf{M}(\mathbf{x})] \cdot \mathbf{v}_i = \mathbf{0},$$

- supremum calculation of the *i-th* eigenvalue

$$\overline{\lambda}_i(\mathbf{x}_{OPT}) \rightarrow \text{maximize value of } \lambda_i \text{ for eq.: } [\mathbf{K}(\mathbf{x}) - \boldsymbol{\lambda}_i \cdot \mathbf{M}(\mathbf{x})] \cdot \mathbf{v}_i = \mathbf{0}.$$

It should be noted that it is possible to realize the searching process by a comparison optimizing method (e.g. Nelder-Mead simplex algorithm) or by using genetic algorithm as a robust tool of global optimization.

5. SOLVING OF TRUSS STRUCTURES WITH INTERVAL PARAMETERS

For the following research purposes on the interval finite element model computing, the truss structure shown on the Figure 7 was analyzed; this figure presents also the geometry of the structure. The truss structure was loaded by uncertain forces F_1 and F_2. The certain model parameters are defined as follows:

- element mass density $\rho = 2700\,\mathrm{kg \cdot m^{-3}}$,
- damping coefficient $\delta = 1{,}3 \cdot 10^{-5}$,
- nodal concentrated mass $m = 50\ \mathrm{kg}$,

(mass point in nodes 33, 34, 35, 36).

Stress interval analysis of the truss structure

Let's assume the following uncertain model parameters:

- Young's modulus $E = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 2 \cdot 10^{11}\,\mathrm{Pa}$,
- cross section areas $A_1 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 569\,\mathrm{mm^2}$,

$A_2 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 691\,\mathrm{mm^2}$,

$A_3 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 1550\,\mathrm{mm^2}$,

- loading forces F_1 and F_2 $F = \langle 0{,}9\ \ 1{,}1 \rangle \cdot \begin{Bmatrix} 35000 \\ 50000 \end{Bmatrix} \mathrm{N}$.

The uncertain input parameters in the vector form are defined for the further analyses as follows: $x = [E, A_1, A_2, A_3, F_1, F_2]$.

The purpose of this study is to compare the efficiency and exactness of the proposed methods MC, COM1, COM2 and OPT. The results of the MC analysis are considered as the reference values and are used for the construction of the solution map. In the case of MC method, 5000 random inputs have been generated; they have been evaluated and properly processed to inf-sup solutions.

The maximal stress values calculated by the particular methods for the truss loaded by maximal stress (truss No. 15) achieved the following values:

$$\max|\sigma_i| = |\sigma_{15}| = \begin{matrix} \langle -343\ \ -255 \rangle & \text{(MC)} \\ \langle -310\ \ -281 \rangle & \text{(COM1)} \\ \langle -343\ \ -254 \rangle & \text{(OPT)} \\ \langle -334\ \ -265 \rangle & \text{(COM2)} \end{matrix}$$

The results are obtained from the final arrangement of the solution set applying the searching algorithm for the infimum and supremum as follows:

$\inf = \min(\max|\sigma(x)|), \quad \sup = \max(\max|\sigma(x)|).$

If the COM1, COM2 and OPT methods are compared with the MC method, it can be observed that:

- the COM1 method is the less appropriate and is not recommended for this kind of analyses,
- the OPT method provides comparable, in some cases even better results than the MC method and what is very important that it does not need so many analyses steps as the MC method,
- the disadvantage of MC and OPT methods is a problem with finding the solution in the solution map corners,
- the COM2 method does not necessarily have to give exact results, but from the perspective of the number of performed analyses, it is more efficient than the MC or OPT methods and it can "find" the solutions in the solution map corners,
- the previous considerations lead to the recommendation to combine COM2 and MC or OPT methods.

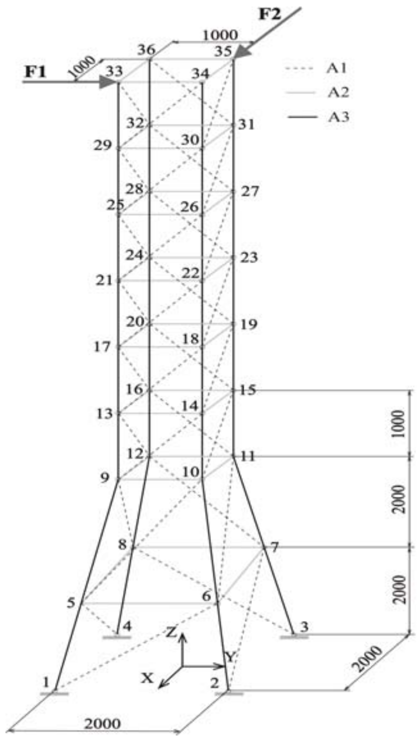

Figure 7. Analyzed truss structure

Table 5. Errors in midpoints of the stress in critical element No. 15

Stress	MC	COM1		OPT		COM2	
	Reference midpoint	Midpoint	Error [%]	Midpoint	Error [%]	Midpoint	Error [%]
σ_{15} [MPa]	-299	-299,5	0,17	-299,5	0,17	-298,5	0,17

Table 6. Errors in radiuses of the stress in critical element No. 15

Stress	MC	COM1		OPT		COM2	
	Reference radius	Radius	Error [%]	Radius	Error [%]	Radius	Error [%]
σ_{15} [MPa]	44	14,5	67	34,5	21,6	44,5	1,1
Application possibility	good	bad		limited		good	

Modal and spectral interval analysis of the truss structure

The interval modal and spectral analysis of the identical truss structure (Figure 7) is realized assuming the following uncertain model parameters:

- Young's modulus $E = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 2 \cdot 10^{11}\,\mathrm{Pa}$,
- cross section areas $A_1 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 569\,\mathrm{mm}^2$,

$A_2 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 691\,\mathrm{mm}^2$,

$A_3 = \langle 0{,}95\ \ 1{,}05 \rangle \cdot 1550\,\mathrm{mm}^2$.

The uncertain input parameters in the vector form are defined for the further analyses as follows: $x = [E, A_1, A_2, A_3]$.

The solution will now consider only the analyses of the first two interval natural frequencies f_1 and f_2. In the case of MC method, 5000 random inputs have been generated, evaluated and properly processed to inf-sup solutions.

The interval solution results are summarized in the Table 7 and the graphical representation of the solving map with the infimum and supremum implementation obtained by using the suggested methods is shown on the Figure 8.

Table 7. Results of the natural frequencies

Freq. no.	MC	COM1	OPT	COM2
f_1 [Hz]	<750 848>	<781 821>	<748 856>	<766 834>
f_2 [Hz]	<770 862>	<797 836>	<769 867>	<784 851>

Table 8. Errors in midpoints of the natural frequencies

Freq. no.	MC	COM1		OPT		COM2	
	Reference midpoint	Midpoint	Error [%]	Midpoint	Error [%]	Midpoint	Error [%]
f_1 [Hz]	799	801	0,25	802	0,375	800	0,125
f_2 [Hz]	816	816,5	0,06	818	0,245	817,5	0,184

Table 9. Errors in radiuses of the natural frequencies

Freq. no.	MC	COM1		OPT		COM2	
	Reference radius	Radius	Error [%]	Radius	Error [%]	Radius	Error [%]
f_1 [Hz]	49	20	59,18	54	10,20	34	30,61
f_2 [Hz]	46	19,5	57,61	49	6,52	33,5	27,17
Application possibility	good	bad		good		limited	

On the basis of the experiences obtained from the interval estimations of the FEM models spectral properties it is possible to conclude:

- the appropriateness of OPT algorithm application, which mainly due to the simplicity of the criteria function for the infimum or supremum analysis gives excellent results, in some cases even better than MC method,
- the previous fact relates to the application of the genetic searching algorithms, of which the biggest advantage is their universality and particularly searching for global extremes,
- the inappropriateness of the COM1 method is demonstrated again because it shows a considerable deflection against the other methods,
- the COM2 method has a limited use but it can be suitable in combination with the OPT method because of "locating" of the solution map corner solutions.

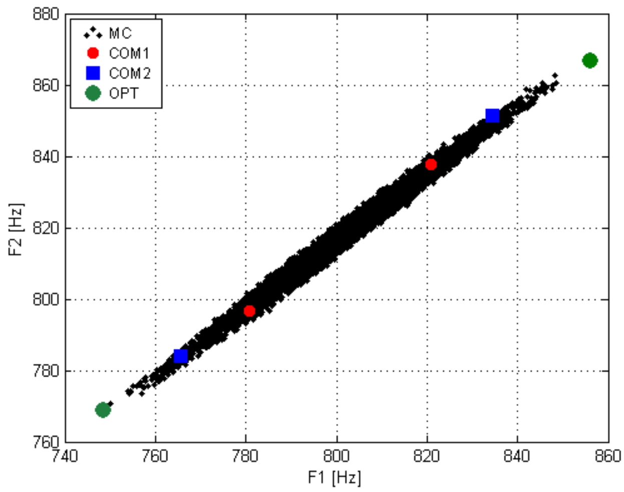

Figure 8. The frequencies solving map with the inf-sup implementation

6. CONCLUSION

The paper discusses the possibility of the interval arithmetic application in a structural analysis. The use of the interval arithmetic provides a new possibility of the quality and reliability appraisal of analyzed objects. Due to this numerical approach, we can analyze mechanical, technological, service and economic properties of the investigated structures more authentically.

Interval finite element method is a useful tool for engineering problems with uncertain – inexact parameters. In the paper we have investigated possibilities of the stress-strain and modal-spectral solution of a truss structure with an interval loading, interval geometry (interval cross section areas of the truss structure) and also interval material properties. We have analyzed the interval stress response and interval natural frequency of the testing FE model. The centre of our interest has been mainly the comparison of the suggested numerical algorithms and their efficiency evaluation.

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