SOME GENERALISATION OF THE BARTLE, DUNFORD
AND SCHWARTZ INTEGRABILITY MODEL

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ABSTRACT
In [4] the author introduces the notion of pseudo submeasure as generalization of the submeasure
concept [2], and studies some properties of the pseudosubmeasure functions with values in a
pseudometric space.
The purpose of this paper is to develop an integration theory for these functions, with respect to a
semigroup valued measure, using families of pseudosubmeasure and the associated topological rings.
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1. PRELIMINARIES

The notions and the notations used here follow the paper [4].
Let \( D \) be an ordered set with the smallest element \( d_0 \). On this set we define a mapping:
\[(d_1, d_2) \to d_1 + d_2 \]
with the following properties:
(P1) \( d_0 + d = d + d_0 \); \( \forall d \in D \)
(P2) \( d_1 + d_2 = d_2 + d_1 \); \( \forall d_1, d_2 \in D \)
(P3) \( d_1 \leq d_2 \Rightarrow d + d_1 \leq d + d_2 \); \( \forall d \in D \)

There exists a subset \( D_1 \subseteq D \) left directed such that
(P4) \( \forall d \in D_1, \exists d \in D \) so that \( d_1 + d \leq d \).

Definition 1.1. A pseudometric on a set \( X \) is a \( D \)-valued function \( p: X \times X \to D \) so that:
(i) \( p(x, y) = d \) \( \iff \) \( x = y \)
(ii) \( p(x, y) = p(y, x) \)
(iii) \( p(x, y) \leq p(x, z) + p(z, y) \); \( x, y, z \in X \).

A set \( X \) together with a pseudometric \( \rho \) is called a pseudometric space and is denoted by
\((X, \rho, D)\).

Remark 1.2. Every uniform space \((X, \mathcal{U})\) is pseudosemimetrizable, [4].

Let \( S \) be a ring (or algebra) of subsets of fixed set \( S \).

Definition 1.3. A pseudosubmeasure on a ring \( S \subseteq \mathcal{P}(S) \) is a mapping \( \gamma: S \to D \) such that:
(S1) \( \gamma(\emptyset) = d_0 \)
(S2) \( E \subseteq F \Rightarrow \gamma(E) \leq \gamma(F) \), \( E, F \in S \)
(S3) \( \gamma(E \cup F) \leq \gamma(E) + \gamma(F) \), \( E, F \in S \)

If \( \gamma \) has the property that \( \gamma(A) = d_0 \Rightarrow A = \emptyset \), then mapping \( p: S \times S \to D \);
\( \rho(A, B) = p(A \Delta B) \) is a pseudometric on \( S \) invariant to translation \( \Delta \) (symmetric difference).

Let \( \Gamma = \{ \gamma_i : S \to D \}_{i \in I} \) be a family of pseudosubmeasure on \( S \subset \mathcal{P}(S) \) and consider the
family \( \Omega_\tau = \{ V_{K, d} : K = \text{finite} \subseteq I, d \in D \} \), where \( V_{K, d} = \{ A \in S : \gamma_i(A) \leq d, A \in K \} \).

Then there exist a \( \tau \)-topology \( \tau(\Gamma) \) on \( S \) so that \( S(\Gamma) = (S, \Delta, \cap, \tau(\Gamma)) \) is a topological ring. Let
\((X, \rho, D)\) be a pseudometric space.
By generalizing the model established in [3], we introduce an uniform structure on $X^S$ in the following way: To every $D_d \in \mathcal{D}$, we associate the set:

$$\mathcal{W}_k(D) = \{(f, g) \in X^S \times X^S ; s \in S, p(f(s), g(s)) \geq d, i \in K\}$$

Then, the family $\{W_k(d); d \in D, K = finite \subset I\}$ forms a base for an uniform structure $\mathcal{U}$ on $X^S$. We denote $X^S(\Gamma) = (X^S, \mathcal{U}_\Gamma)$. The map $f \in X^S$ is a $S$-step function if there exists

$$x_i \in X, E_i \in S, i = 1, 2, ..., n$$

$$x_i \neq x_j, E_i \cap E_j = \emptyset, i \neq j, S = \bigcup_{i=1}^n E_i$$

so that $\forall s \in E_i$ imply $f(s) = x_i, i = 1, 2, ..., n$.

The space of $S$-step functions will be denoted by $E(S, X)$.

**Definition 1.4.** The function $f \in X^S$ is $\Gamma$-pseudosubmeasurable if $f$ belongs to the closure of $E(S, X)$ in $X^S(\Gamma)$.

We denote by $M[S, \Gamma, X]$ the set of these functions.

**Definition 1.5.** Let $\{f_a\}$ be a generalized sequence in $M[S, \Gamma, X]$ and $f \in M[S, \Gamma, X]$. If $f_a \rightharpoonup f$ in $X^S(\Gamma)$, then $\{f_a\}$ converges to $f$ in $\Gamma$-pseudomeasures and we denote $f_a \rightharpoonup f$.

### 2. BASIC ASSUMPTIONS

Let $S$ be a nonempty set, $S \subset P(S)$ be an algebra of subsets of $S$ and consider a family of pseudosubmeasures $\Gamma = \{\gamma_d : S \to D\}_{d \in D^1}$. Let $(X_1, \rho_1, D^1)_i = 1, 2, 3$ be three pseudometric abelian semigroups for which the addition is uniformly continous with respect to the pseudometric $\rho_1$.

In the sequel we consider an additive set function $\mu : S \to X_2, \mu(\emptyset) = 0$, and we will choose a family of pseudosubmeasures as it will be specified. The maps which are to be integrated with respect to $\mu$ will belong to $X_1^S$ and the integral with take values in $X_3$ or its completion $\hat{X}_3$.

Suppose that a separate continuous bilinear map exists $X_1 \times X_2 \to X_3, (x, y) \mapsto xy$ so that:

i) $x \cdot 0 = 0 = y \cdot 0, (x \in X_1, y \in X_2)$

ii) $(x_1 \cdot x_2) \cdot (y_1 \cdot y_2) = x_1 \cdot y_1 \cdot x_2 \cdot y_2 + x_1 \cdot y_1 \cdot x_2 \cdot y_2, (x_1, y_1 \in X_1, x_2, y_2 \in X_2)$.

Finally we suppose that $\Gamma_{\mu, X}$ and the above bilinear map are chosen so that the following continuity axioms are satisfied:

C1) For any $F \in S$ and every $d' \in D^1_3$ there exists $d' \in D^1_3$ with the following property: for any $n \in N$, if $\rho_1(x_i, y_i) < d', i = 1, 2, ..., n$ and $\{E_i\}$ is sequence of pairwise disjoint set from $S$ then:

$$\rho_2 \left( \sum_{i=1}^n x_i \mu(E_i \cap F), \sum_{i=1}^n y_i \mu(E_i \cap F) \right) < d.$$ 

C2) For any $x \in X_1, \lim_{E \to \emptyset} \sum_{E \in S} x \mu(E) = 0$.

### 3. INTEGRABLE FUNCTIONS

Let $f \in E(S, X)$ be a $S$-step function.

**Definition 3.1.** For $E \in S$, the integral of $f$ on $E$ is by definition

$$\int_E f d\mu = \sum_{i=1}^n x_i \mu(E_i \cap E).$$

We denote by $E(S, \Gamma_{\mu}, X_1, X_3)$ the set of $\Gamma_{\mu}$-integrable step functions.
Theorem 3.2. (i) Relatively to the operation \((f + g)(s) = f(s) + g(s)\), the space \(E(S, \Gamma, X_1, X_3)\) is a subsemigroup of \(X_1^3\).

(ii) For \(E \in S\), the map \(f \to \int_E f d\mu\) from \(E(S, \Gamma, X_1, X_3)\) to \(X_3\) is additive.

(iii) For \(f \in E(S, \Gamma, X_1, X_3)\) the map \(E \to \nu(E), \nu(E) = \int_E f d\mu, E \in S\) is an additive function.

(iv) For \(f \in E(S, \Gamma, X_1, X_3)\); \(\lim_{E \to E_0} \nu(E) = \lim_{E \to E_0} \int_E f d\mu = 0\)

The proof follows from definition 3.1 and axioms C_1 and C_2. The extension of the integral from step functions to the arbitrary functions in \(X_1^S\) is based on the following result:

Lemma 3.3. Let \(\{f_a\}\) be a generalized sequence from \(f \in E(S, \Gamma, X_1, X_3)\), which is Cauchy in \(X_1^S(\Gamma)\). For \(\left\{\int_E f_a d\mu\right\}\) to be a Cauchy sequence in \(X_3\) uniform with respect to \(E \in S\) it is necessary and sufficient that:

a) For any neighbourhood \(V\) of 0 in \(X_3\) there exists an index \(\alpha_0, K = \text{finite} \subset I\) and \(d \in D\), so that:\(\alpha \geq \alpha_0\) and \(\gamma_i(E) < d, i \in K\) imply \(\int_E f_a d\mu \in V\)

b) For any neighbourhood \(V\) of 0 in \(X_3\) there exists index \(\alpha_0\) and \(F \in S\) so that \(\int_E f_a d\mu \in V\) if \(\alpha \geq \alpha_0\) and \(E \in S, E \subset S - F\).

Proof. Necessity. For any neighbourhood \(V\) of 0 in \(X_3\) there exists a symmetric entourage \(W\) of the uniform structure from \(X_3\) so that \(W^2(0) \subseteq V\).

Let \(\alpha_0\) be so that \(\left\{\int_E f_a d\mu, \int_E f_a d\mu\right\} \in W\) for any \(E \in S\) if \(\alpha \geq \alpha_0\).

From Theorem 3.2., IV, it results that exists \(d \in D, K = \text{finite} \subset I\) so that we have:

\(\int_E f_a d\mu \in W(0)\) if \(\gamma_i(E) < d, i \in K\). Therefore \(\int_E f_a d\mu \in V\) if \(\alpha \geq \alpha_0\) and \(\gamma_i(E) < d, i \in K\), that is the condition a). The condition b) is obtained by taking \(E = \{s \in S : f_a(s) \neq 0\}\). We have \(F \in S\), and \(\int_E f_a d\mu = 0\) for all \(E \in S\) with \(E \subset S - F\).

Sufficiency. Let \(W\) be a symmetric entourage for \(X_3\) and let \(\alpha_0, K = \text{finite} \subset I\), \(d \in D\) and \(F\) be chosen depending on the neighbourhood \(W(0)\) according to the conditions a) and b) simultaneously. For \(F\) and \(W\), let entourage \(U\) from \(X_1\) be chosen according to axiom \(C_1\).

We write: \(F_{aa} = \{s \in S : (f_a(s), f_a(s)) \notin U\}^1, F_{aa} \in S\).

Since \(\{f_a\}\) is Cauchy in \(X_1^S(\Gamma)\) there exists \(\alpha_i \geq \alpha_0\) so that \(\gamma_i(F_{aa}) < d, i \in K\) for \(\alpha, \alpha' \geq \alpha_i\). For \(E \in S\) in the semigroup \(X_1 \times X_3\), we can write:

\[
\left(\int_E f_a d\mu, \int_E f_a d\mu\right) = \left(\int_{E \cap F_{aa}} f_a d\mu, \int_{E \cap F_{aa}} f_a d\mu\right) + \left(\int_{E \cap F_{aa}} f_a d\mu, \int_{E \cap F_{aa}} f_a d\mu\right) + \left(\int_{E \cap F_{aa}} f_a d\mu, \int_{E \cap F_{aa}} f_a d\mu\right) + \left(\int_{E \cap F_{aa}} f_a d\mu, \int_{E \cap F_{aa}} f_a d\mu\right) \subset W(0) \times W(0) + W(0) \times W(0) + W \subseteq W^2 + W^2 + W^2, \alpha, \alpha' \geq \alpha_i
\]
Corollary 3.4. Let \( \{f_\alpha\} \) and \( \{g_\beta\} \) be two generalized sequences from \( E(S, \Gamma_\mu, X_1, X_3) \), convergent in \( X_1^S(\Gamma_\mu) \) to the same function.

If \( \int_E f_\alpha d\mu_E \) and \( \int_E g_\beta d\mu_E \) are generalized Cauchy sequences in \( X_3 \) uniformly in \( E \in S \), then for any entourage \( W \) from \( X_3 \) there exists \( \alpha_0 \) and \( \beta_0 \) so that if \( \alpha \geq \alpha_0 \), \( \beta \geq \beta_0 \) it results that \( \left( \int_E f_\alpha d\mu, \int_E g_\beta d\mu \right) \in W \), uniformly in \( E \in S \).

Proof. Given a symmetric entourage \( W_i \) from \( X_1 \) so that \( W_i^2 + W_i^2 + W_i^2 \subseteq W \) we choose an entourage \( U \) from \( X_1 \) corresponding to \( W_i \) according to axiom \( C_1 \).

We write \( F_{af} = \{ \delta \in S; (f_\alpha(s), g_\beta(s)) \notin U \} \). From the previous Lemma it results that there exits \( \alpha_0, \beta_0, d \in D, K = \text{finite} \subseteq I \) so that if \( F \in S \) and \( \alpha > \alpha_0, \beta > \beta_0, \gamma_i(E) < d, i \in K, E \subseteq S - F, E \in S \) we have \( \int_E f_\alpha d\mu \in W_i(0) \) and \( \int_E g_\beta d\mu \in W_i(0) \).

By hypothesis there exist if \( \alpha_i \geq \alpha_0 \) and \( \beta_i \geq \beta_0 \) so that for \( \alpha > \alpha_1, \beta > \beta_1 \), we have \( \gamma_i(F_{af}) < d, i \in K \). Expressing the pair \( \left( \int_E f_\alpha d\mu, \int_E g_\beta d\mu \right) \) in the same way as in the proof of the sufficiency from Lemma 3.3., the result is obtained.

Definition 3.5. The function \( f \in X_1^S \) is called \( \Gamma_\mu \)-integrable of there exists a generalized sequence \( \{f_\alpha \text{ from } E(S, \Gamma_\mu, X_1, X_3)\} \) so that \( f_\alpha \xrightarrow{\Gamma_\mu} f \) and \( \left\{ \int_E f_\alpha d\mu \right\} \) is a generalized Cauchy sequence in \( X_3 \), uniformly in \( E \in S \). Then the \( \Gamma_\mu \)-integral is the element from \( \hat{X}_3 \) the completion of \( X_3 \), defined by: \( \left[ \int_E f_\alpha d\mu = \lim_{\alpha} \int_E f_\alpha d\mu \right. \).

From the Corollary 3.4 it results that above \( \Gamma_\mu \)-integral is properly defined. We denote by \( L(S, \Gamma_\mu, X_1, X_3) \) the set of \( \Gamma_\mu \)-integrable functions from \( M[S, \Gamma_\mu, X_1] \).

It is obvious that \( E(S, \Gamma_\mu, X_1, X_3) \subseteq L(S, \Gamma_\mu, X_1, X_3) \) and the \( \Gamma_\mu \)-integral restricted to \( E(S, \Gamma_\mu, X_1, X_3) \) coincides with the \( \Gamma_\mu \)-integral from Definition 3.1.

Theorem 3.6. Relatively to the operation of addition the set \( L(S, \Gamma_\mu, X_1, X_3) \) is a subsemigroup of \( X_1^S \)

(i) For \( E \in S \), the mapping \( f \mapsto \int_E fd\mu \) of \( L(S, \Gamma_\mu, X_1, X_3) \) in \( \hat{X}_3 \) is additive:

\[ \int_E (f + g)d\mu = \int_E fd\mu + \int_E gd\mu, \text{ for } f, g \in L(S, \Gamma_\mu, X_1, X_3) \]

(ii) For \( f \in L(S, \Gamma_\mu, X_1, X_3) \) the mapping \( E \mapsto v(E) = \int_E fd\mu \) is additive:

\[ v\left( \bigcup_{i=1}^n E_i \right) = \sum_{i=1}^n v(E_i), \text{ for } E_i \cap E_j = \emptyset, i \neq j, v(\emptyset) = 0 \]

(iii) For \( f \in L(S, \Gamma_\mu, X_1, X_3) \) we have:

\[ \lim_{E \in S} v(E) = 0 \]

The proof follows from Corollary 3.4. and the definition 3.5.
REFERENCES