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SOME GENERALISATION OF THE BARTLE, DUNFORD AND SCHWARTZ INTEGRABILITY MODEL

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ABSTRACT
In [4] the author introduces the notion of pseudosubmeasure as generalization of the submeasure concept [2], and studies some proprieties of the pseudosubmeasure functions with values in a pseudometric space.
The purpose of this paper is to develop an integration theory for these functions, with respect to a semigroup valued measure, using families of pseudosubmeasure and the associated topological rings.
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1. PRELIMINARIES

The notions and the notations used here follow the paper [4].

Let D be an ordered set with the smallest element d_0 .On this set we define a mapping: $(d_1, d_2) \to d_1 + d_2$ with the following properties:

(P1) $d_o + d = d + d_0; \forall d \in D$

(P2) $d_1 + d_2 = d_2 + d_1; \forall d_1, d_2 \in D$

(P3) $d_1 \le d_2 \Rightarrow d + d_1 \le d + d_2; \forall d \in D$

There exists a subset $D_1 \subseteq D$ left directed such that

(P4) $\forall d \in D_1, \exists d_1 \in D$ so that $d_1 + d_1 \le d$.

Definition 1.1. *A pseudometric on a set X is a D-valued function* $p : X \times X \to D$ *so that:*

(i) $p(x, y) = d_0 \Leftrightarrow x = y$

(ii) $p(x, y) = p(y, x), x, y, z \in X$

(iii) $p(x, y) \le p(x, z) + p(z, y); x, y, z \in X$.

A set X together with a pseudometric ρ *is called a pseudometric space and is denoted by* (X, ρ, D).

Remark 1.2. *Every uniform space* $(X, \mathcal{U})$ *is pseudosemimetrizable*, [4].

Let $\mathcal{S}$ be a ring (or algebra) of subsets of fixed set *S*.

Definition 1.3. *A pseudosubmeasure on a ring* $\mathcal{S} \subset \mathcal{P}(S)$ *is a mapping* $\gamma : \mathcal{S} \to D$ *such that:*

(S_1) $\gamma(\varnothing) = d_0$

(S_2) $E \subseteq F \Rightarrow \gamma(E) \le \gamma(F), E, F \in \mathcal{S}$

(S_3) $\gamma(E \cup F) \le \gamma(E) + \gamma(F), E, F \in \mathcal{S}$

If γ *has the propert that* $\gamma(A) = d_0 \Rightarrow A = \varnothing$, *then mapping* $p : \mathcal{S} \times \mathcal{S} \to D$; $\rho(A, B) = \rho(A \Delta B)$ *is a pseudometric on* $\mathcal{S}$ *invariant to translation* Δ *(symmetric difference).*

Let $\Gamma = \{\gamma_i : \mathcal{S} \to D\}_{i \in I}$ be a family of pseudosubmeasure on $\mathcal{S} \subset P(S)$ and consider the family $\Omega_\Gamma = \{\nu_{K,d} : K = finite \subseteq I, d \in D_1\}$, where $\nu_{K,d} = (A \in \mathcal{S} : \gamma_i(A) \le d, a \in K\}$.

Then there exist a FN-topology $\tau(\Gamma)$ on $\mathcal{S}$ so that $\mathcal{S}(\Gamma) = (\mathcal{S}, \Delta, \cap, \tau(\Gamma))$ is a topical ring. Let (X, ρ, D) be a pseudometric space.

By generalizing the model established in [3], we introduce an uniform structure on X^S in the following way: To every $K = finite \subset I, d \in D$,we associate the set:

$$\mathcal{W}_k(D) = \{(f,g) \in X^S \times X^S; \gamma_i\{s \in S; \rho(f(s), g(s)) \geq d\} < d, i \in K\}$$

Then, the family $\{W_k(d); d \in D_1, K = finite \subset I\}$ forms a base for an uniform structure $\mathcal{U}_\Gamma$ on X^S. We denote $X^S(\Gamma) = (X^S, \mathcal{U}_\Gamma)$. The map $f \in X^S$ is a $\mathcal{S}$-step function if there exists $x_i \in X, E_i \in \mathcal{S}, i = 1,2,...,n$ $x_i \neq x_j, E_i \cap E_j = \emptyset, i \neq j, S = \bigcup_{i=1}^{n} E_i$ so that $\forall s \in E_i$ imply $f(s) = x_i, i = 1,2,...,n.$

The space of $\mathcal{S}$-step functions will be denoted by $\mathcal{E}(\mathcal{S}, X)$.

Definition 1.4. *The function $f \in X^S$ is Γ - pseudosubmeasurable iff f belongs to the closure of $\mathcal{E}(\mathcal{S}, X)$ in $X^S(\Gamma)$.*

We denote by $\mathcal{M}[\mathcal{S}, \Gamma, X]$ the set of these functions.

Definition 1.5. *Let $\{f_a\}$ be a generalized sequence in $\mathcal{M}[\mathcal{S}, \Gamma, X]$ and $f \in \mathcal{M}[\mathcal{S}, \Gamma, X]$. If $f_a \to f$ in $X^S(\Gamma)$, then $\{f_a\}$ converges to f in Γ - pseudomeasures and we denote $f_a \xrightarrow{r} f$.*

2. BASIC ASSUMPTIONS

Let S be a nonempty set, $\mathcal{S} \subset P(S)$ be an algebra of subsets of S and consider a family of pseudosubmeasures $\Gamma = \{\gamma_i : \mathcal{S} \to D\}_{i \in I}$.

Let $(X_i, \rho_i, D^i), i = 1,2,3$ be three pseudometric abelian semigroups for which the addition is uniformly continous with respect to the pseudometric ρ_i).

In the sequel we consider an additive set function $\mu : \mathcal{S} \to X_2, \mu(\emptyset) = 0$, and we will choose a family of pseudosubmeasures as it will be specified. The maps which are to be integrated with respect to μ will belong to X_1^S and the integral with take values in X_3 or its completion $\hat{X}_3$.

Suppose that a separate continuous bilinear map exists $X_1 \times X_2 \to X_3; (x,y) \mapsto x \cdot y$ so that:

i) $x \cdot 0 = 0 \cdot y = 0, (x \in X_1, y \in X_2)$

ii) $(x_1 + x_2) \cdot (y_1 + y_2) = x_1 \cdot y_1 + x_1 \cdot y_2 + x_2 \cdot y_1 + x_2 \cdot y_2, (x_1, y_1 \in X_1, x_2, y_2 \in X_2).$

Finally we suppose that Γ_μ, μ and the above bilinear map are chosen so that the following continuity axioms are satisfied:

C1) For avery $F \in \mathcal{S}$ and every $d' \in D_1^3$ there exists $d' \in D_1^1$ with the following property: for any $n \in N$, if $\rho_1(x_i, y_i) < d', i = 1,2,...,n$ and $\{E_i\}$ is sequence of pairwise disjoint set from $\mathcal{S}$ then: $\rho_3\left(\sum_{i=1}^{n} x_i \mu(E_i \cap F), \sum_{i=1}^{n} y_i \mu(E_i \cap F)\right) < d.$

C2) For any $x \in X_1$, $\lim_{\substack{E \to \emptyset \\ E \in \mathcal{S}}} x\mu(E) = 0.$

3. INTEGRABLE FUNCTIONS

Let $f \in \mathcal{E}(\mathcal{S}, X)$ be a $\mathcal{S}$-step function.

Definition 3.1. *For E $\in \mathcal{S}$, the integral of f on E is by definition* $\int_E f d\mu = \sum_{i=1}^{n} x_i \mu(E_i \cap E).$

We denote by $\mathcal{E}(\mathcal{S}, \Gamma_\mu, X_1, X_3)$ the set of Γ_μ-integrable step functions.

Theorem 3.2. (i) *Relatively to the operation* $(f+g)(s)=f(s)+g(s)$, *the space* $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *is a subsemigroup of* X_1^3.

(ii) *For* $E\in S$ *, the map* $f\to\int_E fd\mu$ *from* $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *to* X_3 *is additive.*

(iii) *For* $f\in\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *the map* $E\to v(E), v(E)=\int_E fd\mu, E\in\mathcal{S}$ *is an additive function.*

(iv) *For* $f\in\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$; $\lim\limits_{E\xrightarrow{\Gamma_\mu}0} v(E)=\lim\limits_{E\xrightarrow{\Gamma_\mu}0}\int_E fd\mu=0$

The proof follows from definition 3.1 and axioms C_1 and C_2. The extension of the integral from step functions to the arbitrary functions in X_1^S is based on the following result:

Lemma 3.3. *Let* $\{f_a\}$ *be a generalized sequence from* $f\in\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$, *which is Cauchy in* $X_1^S(\Gamma_\mu)$. *For* $\left\{\int_E f_\alpha d\mu\right\}$ *to be a Cauchy sequence in* X_3 *uniform with respect to* $E\in\mathcal{S}$ *it is necessary and sufficient that:*

a) For any neightbourhood V of 0 in X_3 *there exists an index* $\alpha_0, K=finite\subset I$ *and* $d\in D$, *so that* $:\alpha\ge\alpha_0$ *and* $\gamma_i(E)<d, i\in K$ *imply* $\int_E f_\alpha d\mu\in V$

b) For sny neighbourhood V of 0 in X_3 *there exists and index* α_0 *and* $F\in\mathcal{S}$ *so that* $\int_E f_\alpha d\mu\in V$ *if* $\alpha\ge\alpha_0$ *and* $E\in\mathcal{S}, E\subset S-F$.

Proof. Necessity. For any neighbourhood *V* of 0 in X_3 there exists a symmetric entourage *W* of the uniform structure from X_3 so that $W^2(0)\subseteq V$.

Let α_0 be so that $\left(\int_E f_\alpha d\mu,\int_E f_{\alpha_0}d\mu\right)\in W$ for any $E\in\mathcal{S}$ if $\alpha\ge\alpha_0$.

From Theorem 3.2., IV, it results that exists $d\in D_1, K=finite\subset I$ so that we have: $\int_E f_{\alpha_0}d\mu\in W(0)$ if $\gamma_i(E)<d, i\in K$. Therefore $\int_E f_\alpha d\mu\in V$ if $\alpha\ge\alpha_0$ and $\gamma_i(E)<d, i\in K$, that is the condition a). The condition b) is obtained by taking $E=\{s\in S: f_{\alpha_0}(s)\ne 0\}$. We have $F\in\mathcal{S}$, and $\int_E f_{\alpha_0}d\mu=0$ for all $E\in\mathcal{S}$ with $E\subset S-F$.

Sufficiency. Let *W* be a symmetric entourage for X_3 and let $\alpha_0, K=finite\subset I$, $d\in D_1$ and *F* be chosen depending on the neightbourhood *W*(0) according to the conditions a) and b) simultaneously. For *F* and *W*, let entourage *U* from X_1 be chosen according to axiom C_1.

We write: $F_{\alpha\alpha'}=\{s\in S;(f_\alpha(s),f_{\alpha'}(s))\notin U\}, F_{\alpha\alpha'}\in S.$

Since $\{f_\alpha$ is Cauchy in $X_1^S(\Gamma_\mu)$ there exists $\alpha_1\ge\alpha_0$ so that $\gamma_i(F_{\alpha\alpha'})<d, i\in K$ for $\alpha,\alpha'\ge\alpha_1$. For $E\in\mathcal{S}$ in the semigroup $X_3\times X_3$, we can write:

$$\left(\int_E f_\alpha d\mu,\int_E f_{\alpha'}d\mu\right)=\left(\int_{E\cap F_{\alpha\alpha'}} f_\alpha d\mu,\int_{E\cap F_{\alpha\alpha'}} f_{\alpha'}d\mu\right)+\left(\int_{E\setminus(F_{\alpha\alpha'}\cup F)} f_\alpha d\mu,\int_{E\setminus(F_{\alpha\alpha'}\cup F)} f_{\alpha'}d\mu\right)+$$

$$\left(\int_{E\setminus(F_{\alpha\alpha'}\cap F)} f_\alpha d\mu,\int_{E\setminus(F_{\alpha\alpha'}\cap F)} f_{\alpha'}d\mu\right)\in W(0)\times W(0)+W(o)\times W(0)+W\subseteq W^2+W^2+W^2,\alpha,'\ge\alpha_1$$

Corollary 3.4. *Let* $\{f_\alpha\}$ and $\{g_\beta\}$ *be two generalized sequences from* $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$, *convergent in* $X_1^S(\Gamma_\mu)$ *to the same function.*

If $\left\{\int_E f_\alpha d\mu,\right\}$ *and* $\left\{\int_E g_\beta d\mu\right\}$ *are generalized Cauchy sequences in* X_3 *uniformly in* $E\in\mathcal{S}$, *then for any entourage W from* X_3 *there exists* α_0 *and* β_0 *so that if* $\alpha\ge\alpha_0$, $\beta\ge\beta_0$ *it results that* $\left(\int_E f_\alpha d\mu,\int_E g_\beta d\mu\right)\in W$ *, uniformly in* $E\in\mathcal{S}$.

Proof. Given a symmetric entourage W_1 from X_3 so that $W_1^2+W_1^2+W_1^2\subseteq W$ we choose an entourage U from X_1 corresponding to W_1 according to axiom C_1.

We write $F_{\alpha\beta}=\{s\in S;(f_\alpha(s),g_\beta(s))\notin U\}$. From the previous Lemma it results that there exits $\alpha_0,\beta_0,d\in D,K=\mathit{finite}\subset I$ so that if $F\in\mathcal{S}$ and $\alpha>\alpha_0,\beta>\beta_0,\gamma_i(E)<d,i\in K$, $E\subset S-F$, $E\in\mathcal{S}$ we have $\int_E f_\alpha d\mu\in W_1(0)$ and $\int_E f_\beta d\mu\in W_1(0)$

By hypothesis there exist if $\alpha_1\ge\alpha_0$ and $\beta_1\ge\beta_0$ so that for $\alpha>\alpha_1,\beta>\beta_1$, we have $\gamma_i(F_{\alpha\beta})<d,i\in K$. Expressing the pair $\left(\int_E f_\alpha d\mu,\int_E g_\beta d\mu\right)$ in the same way as in the proof of the sufficiency from Lemma 3.3., the result is obtained.

Definition 3.5. *The function* $f\in X_1^S$ *is called* Γ_μ*- integrable of there exists a generalized sequence* $\{f_\alpha$ *from* $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)\}$ *so that* $f_\alpha\xrightarrow{\Gamma_\mu}f$ *and* $\left\{\int_E f_\alpha d\mu,\right\}$ *is a generalized Cauchy sequence in* X_3, *uniformly in* $E\in\mathcal{S}$. *Then the* Γ_μ*-integral is the element from* $\hat{X}_3$ *the completion of* X_3, *defined by:* $\int_E f_\alpha d\mu=\lim_\alpha\int_E f_\alpha d\mu$.

From the Corollary 3.4 it results that above Γ_μ-integral is properly defined. We denote by $\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ the set of Γ_μ-integrable functions from $\mathcal{M}\lfloor\mathcal{S},\Gamma_\mu,X_1\rfloor$.

It is obvious that $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)\subset\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ and the Γ_μ-integral restricted to $\mathcal{E}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ coincides with the Γ_μ-integral from Definition 3.1.

Theorem 3.6. *Relatively to the operation of addition the set* $\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *is a subsemigroup of* X_1^S

(i) *For* $E\in\mathcal{S}$, *the mapping* $f\to\int_E fd\mu$ *of* $\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *in* $\hat{X}_3$ *is additive:*

$$\int_E(f+g)d\mu=\int_E fd\mu+\int_E gd\mu,\ f,g\in\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$$

(ii) *For* $f\in\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *the mapping* $E\to v(E)=\int_E fd\mu,E\in\mathcal{S}$ *is additive:*

$$v\left(\bigcup_{i=1}^n E_i\right)=\sum_{i=1}^n v(E_i),\ E_i\cap E_j=\emptyset,i\ne j,v(\emptyset)=0$$

(iii) *For* $f\in\mathcal{L}(\mathcal{S},\Gamma_\mu,X_1,X_3)$ *we have:* $\lim\limits_{E\xrightarrow[E\in\mathcal{S}]{\Gamma_\mu}} v(E)=0$

The proof follows from Corollary 3.4. and the definition 3.5.

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