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THE BENDING OF THE FINITE ELASTIC ROD ON ELASTIC FOUNDATION UNDER THE ACTION OF CONCENTRATED LOADS

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Abstract
The paper gives solution in the distributions space D'_+ for the boundary-value problems regarding the bending of the elastic rods on elastic foundation. The expression of the rod deflection is given with the help of the fundamental solution in D'_+ of the operator which describes the rod bending. We admit that on the rod act uniformly distributed loads of intensity q, as well as a concentrated load of value P. We have considered a straight homogeneous elastic rod of finite length ℓ, and with constant cross-section, supported in point O and with elastic fixing in point A. The rod lies on an elastic foundation of Winkler type. The deflection of the rod as well as the reactions in the rod ends is given.

1. INTRODUCTION

In solving the problem of the bending of the finite elastic rod on elastic foundation under the action of concentrated loads we come across difficulties owing to these concentrated loads and moments.

The general and unitary method to deal with the problems concerning discontinuous loading is the distribution theory. In the framework of this theory a single equation which contains the boundary and jump conditions is obtained.

The distribution theory was used in [2], [6], [7], [8] and [9] for analyzing beams with discontinuities. A bending problem with discontinuities in which the distribution theory isn't systematically applied, being a combination between classical mathematical analysis and the distribution theory, is studied in [10]. In [2] and [6] using the distribution theory in a systematic manner we study the bending problem with discontinuities of a finite elastic rod on elastic foundation under the action of concentrated loads.

In this paper we study the bending of a straight homogeneous elastic rod of finite length, with constant cross-section, supported in the left end and with elastic fixing in the right end, which lies on an elastic foundation of Winkler type.

We determine the deflection of the rod as well as the reactions of the rod ends. The obtained result allows a global analysis of the influence of the each term: support, elastic fixing and the concentrated load.

2. THE STUDY OF THE BENDING OF THE FINITE ELASTIC ROD

Let be OA a straight homogeneous elastic rod of finite length ℓ, and with constant cross-section, supported in point O and with elastic fixing in point A, which lies on an elastic foundation of Winkler type [1]. We admit that on the rod act uniformly distributed loads of intensity q, as well as a concentrated load of value P applied in point $c \in (0, \ell)$.

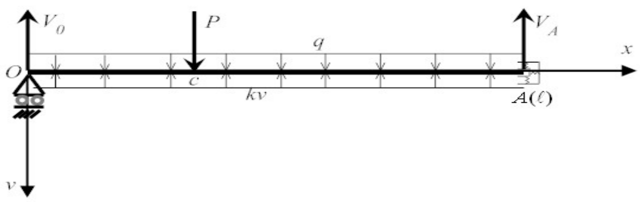

Fig. 2.1. Elastic rod supported on an elastic foundation

We shall denote by $v(x),\ x \in [0, \ell]$ the deflection of the rod. We denote by $\tilde{\partial}_x = \dfrac{\tilde{d}}{dx}$, $\partial_x = \dfrac{d}{dx}$ the derivative in classic sense and the derivative in distribution sense, respectively.

For a Winkler model, it is assumed that the reaction of the elastic foundation $q_e(x),\ x\in[0,\ell]$ exerted on the rod is proportional to its deflection at that point and is independent from the deflection of other parts of the foundation hence.

$$q_e(x)=-kv(x),\qquad x\in[0,\ell], \tag{2.1}$$

where k is called the rigidity coefficient of the elastic foundation.

We shall denote by $D'(\mathbf{R})$ the distribution (continuous linear functional) defined on the test functions space $D'(\mathbf{R})$, which are indefinite derivable functions with compact support.

We denote by $D'_+\subset D'(\mathbf{R})$ the distributions from $D'(\mathbf{R})$ having the supports on $[0,\infty)$. We mention that the distributions from D'_+ represent a convolution algebra without divisors of zero. We observe that $\tilde v(x)=\begin{cases} v(x), & x\in[0,\ell],\\ 0, & x\notin[0,\ell],\end{cases}$ represents a function type distribution from D'_+, because its support is in $[0,\ell]\subset[0,\infty)$.

We denote by the symbol $[\ \ ]_a$ the jump of a certain value at point $x=a$.

Due to the way in which the rod is fixed the boundary conditions are

$$\begin{gathered}\tilde v(0+0)=0,\ \tilde v(0-0)=0,\ \tilde v(\ell+0)=0,\ \tilde v(\ell-0)=0,\\ \tilde\partial_x^2\tilde v(0+0)=0,\tilde\partial_x^2\tilde v(0-0)=0,\ \tilde\partial_x^2\tilde v(\ell+0)=0,\ EI\tilde\partial_x^2\tilde v(\ell-0)=k_1\tilde\partial_x\tilde v(\ell-0).\end{gathered} \tag{2.2}$$

From the boundary conditions (2.2) we have

$$\begin{gathered}\left[\tilde v\right]_0=\tilde v(0+0)-\tilde v(0-0)=0,\ \left[\tilde v\right]_\ell=\tilde v(\ell+0)-\tilde v(\ell-0)=0,\\ \left[\tilde\partial_x\tilde v\right]_0=\tilde v'(0+0)-\tilde v'(0-0)=\tilde v'(0+0),\ \left[\tilde\partial_x\tilde v\right]_\ell=\tilde v'(\ell+0)-\tilde v'(\ell-0)=-\tilde v'(\ell-0).\end{gathered} \tag{2.3}$$

According to [2] for the deflection $\tilde v$ we have the expression

$$\tilde v(x)=\begin{cases} 0, & x\notin[0,\ell)\\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(x-t)dt-\dfrac{V_0}{4EI\omega^3}H(x)u(x)+\dfrac{P}{4EI\omega^3}H(x-c)u(x-c) & \\ +\dfrac{1}{4\omega^3}\tilde v'(0+0)H(x)u_2(x), & x\in[0,\ell]\end{cases} \tag{2.4}$$

where $\omega=\sqrt[4]{\dfrac{k}{4EI}}$. We mention that we introduce the real-valued functions $u,u_1,u_2,u_3\in C^\infty(\mathbf{R})$ having the expression:

$$\begin{aligned} &u(x)=\cosh\omega x\sin\omega x-\sinh\omega x\cos\omega x,\\ &u_1(x)=u'(x)=2\omega\sinh\omega x\sin\omega x,\\ &u_2(x)=u''(x)=2\omega^2\left(\cosh\omega x\sin\omega x+\sinh\omega x\cos\omega x\right),\\ &u_3(x)=u'''(x)=4\omega^3\left(\cosh\omega x\cos\omega x\right).\end{aligned} \tag{2.5}$$

We have $u^4(x)=u_3'(x)=-4\omega^4u(x)$.

From here results

$$u^{(4k)}(x)=\left(-4\omega^4\right)^k u(x),\qquad u^{(4k+1)}(x)=\left(-4\omega^4\right)^k u_1(x),$$

$$u^{(4k+2)}(x)=\left(-4\omega^4\right)^k u_2(x),\qquad u^{(4k+3)}(x)=\left(-4\omega^4\right)^k u_3(x).$$

Because any natural number $n\ge 4$ can be written under the form $n=4k+p,\ p=0,1,2,3;\ k\in\mathbb{N}$, we have:

Any $n\ge 4$ order derivative of the function $u\in C^\infty(\mathbf{R})$ represents a multiple of one of the functions $u, u_1=u'$, $u_2=u''$, $u_3=u'''$ namely

$$u^{(n)}(x) = \begin{cases} \left(-4\omega^4\right)^k u(x), & n = 4k \\ \left(-4\omega^4\right)^k u_1(x) & n = 4k+1 \\ \left(-4\omega^4\right)^k u_2(x) & n = 4k+2 \\ \left(-4\omega^4\right)^k u_3(x) & n = 4k+3 \end{cases} \quad k = 1, 2, 3, \ldots$$

Using the formula $\int_0^x f(x-t)dt = \int_0^x f(t)dt$ the deflection $\tilde{v}$ can be written under the form

$$\tilde{v}(x) = \begin{cases} 0, & x \notin [0,\ell) \\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(t)dt - \dfrac{V_0 u(x)}{4EI\omega^3} + \dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3}, & x \in [0,c) \\ \dfrac{q}{4EI\omega^3}\displaystyle\int_0^x u(t)dt - \dfrac{V_0 u(x)}{4EI\omega^3} + \dfrac{\tilde{v}'(0+0)u_2(x)}{4\omega^3} + \dfrac{Pu(x-c)}{4EI\omega^3}, & x \in [c,\ell] \end{cases} \tag{2.6}$$

We observe that in this relation of the deflection $\tilde{v}$ appear only two unknowns, namely: the reaction V_0 in O and the rotation of rod to the right in point O, $\tilde{v}'(0+0)$. These unknowns as well as the unknowns $V_A, m_A, \tilde{v}'(\ell-0)$ representing the reaction and moment in the A as well as the rotation of rod to the left in point A, respectively, will be determined from the following conditions:

$$q\int_0^\ell u(t)dt - V_0 u(\ell) + Pu(\ell-c) + EI\tilde{v}'(0+0)u_2(\ell) = 0, \tag{2.7}$$

$$q\int_0^\ell u_1(t)dt - V_0 u_1(\ell) + Pu_1(\ell-c) + EI\left[\tilde{v}'(0+0)u_2(\ell) - 4\omega^3\tilde{v}'(\ell-0)\right] = 0, \tag{2.8}$$

$$q\int_0^\ell u_2(t)dt - V_0 u_2(\ell) + Pu_2(\ell-c) + 4\omega^3 m_A - 4\omega^4 EI\tilde{v}'(0+0)u(\ell) = 0, \tag{2.9}$$

$$q\int_0^\ell u_3(t)dt - V_0 u_3(\ell) + Pu_3(\ell-c) - 4\omega^3 V_A - 4\omega^4 EI\tilde{v}'(0+0)u_1(\ell) = 0, \tag{2.10}$$

$$m_A = k_1\tilde{v}'(\ell-0), \tag{2.11}$$

where k_1 represents a proportionality factor.

The relations (2.7)-(2.10) was obtained from the condition that the support of the deflection should be $[0,\ell]$, namely $\operatorname{supp}\tilde{v} = [0,\ell]$.

From the above system of equations we shall obtain the unknowns V_0, $V_A, m_A, \tilde{v}'(\ell-0)$ and $\tilde{v}'(0+0)$.

We have the expression

$$V_0 = \frac{k_1\left(b_1u_2(\ell) - b_0u_3(\ell)\right) + EI\left(b_2u_2(\ell) + 4\omega^4 u(\ell)b_0\right)}{EI\left(u_2^2(\ell) + 4\omega^4u^2(\ell)\right) - k_1\left(u_3(\ell)u(\ell) - u_2(\ell)u_1(\ell)\right)}, \tag{2.12}$$

$$\tilde{v}'(0+0) = \frac{1}{EI}\frac{EI\left(b_2u(\ell) - b_0u_2(\ell)\right) + k_1\left(b_1u(\ell) - b_0u_1(\ell)\right)}{EI\left(u_2^2(\ell) + 4\omega^4u^2(\ell)\right) - k_1\left(u_3(\ell)u(\ell) - u_2(\ell)u_1(\ell)\right)}, \tag{2.13}$$

$$\tilde{v}'(\ell-0) = \frac{u_2(\ell)\left(b_1u_2(\ell) - b_2u_1(\ell)\right) + 4\omega^4u(\ell)\left(b_1u(\ell) - b_0u_1(\ell)\right) + u_3(\ell)\left(b_2u(\ell) - b_0u_2(\ell)\right)}{4\omega^3\left[EI\left(u_2^2(\ell) + 4\omega^4u^2(\ell)\right) - k_1\left(u_3(\ell)u(\ell) - u_2(\ell)u_1(\ell)\right)\right]}, \tag{2.14}$$

$$m_A = k_1\frac{u_2(\ell)\left(b_1u_2(\ell) - b_2u_1(\ell)\right) + 4\omega^4u(\ell)\left(b_1u(\ell) - b_0u_1(\ell)\right) + u_3(\ell)\left(b_2u(\ell) - b_0u_2(\ell)\right)}{4\omega^3\left[EI\left(u_2^2(\ell) + 4\omega^4u^2(\ell)\right) - k_1\left(u_3(\ell)u(\ell) - u_2(\ell)u_1(\ell)\right)\right]}, \tag{2.15}$$

where

$$\begin{aligned} b_0 &= \frac{q}{\omega} - \frac{q}{\omega}\frac{u_3(\ell)}{4\omega^3} + Pu(\ell - c), \\ b_1 &= qu(\ell) + Pu_1(\ell - c), \\ b_2 &= q\omega^2 u_1(\ell) + Pu_2(\ell - c), \\ b_3 &= qu_2(\ell) + Pu_3(\ell - c). \end{aligned} \tag{2.16}$$

3. CONCLUSIONS

As it was pointed out in [2] the distribution theory represents the adequate framework to solve the boundary-value problems regarding the bending of the elastic rods on elastic foundation when we have external discontinuities (e.g. discontinuous loading) and internal discontinuities (e.g. owning to the mechanical properties).

In this way the difference between continuous loads and discontinuous loads is vanish.

The classical method of solving the problems in which appear discontinuities is the partition of the rod into segments (which have distinct mechanical and geometrical properties). We obtain a system of boundary (the ends of the segments rod) value problems so that the solution of the problem on each rod segment is continuous. To solve the problem with discontinuities we must take into account the continuity conditions at the interface of the rod segments.

The obtained result allows a global analysis of the influence of the each term: support, elastic fixing and the concentrated load.

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