ANNALS of Faculty Engineering Hunedoara – International Journal of Engineering
ISSN: 1584 - 2673

NUMERICAL ALGORITHMS FOR SPATIO-TEMPORAL STABILITY OF VISCOUS SWIRLING FLOWS

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ABSTRACT:
This paper presents a methodology to derive the mathematical models and the algorithms for numerical investigation upon the spatio-temporal stability of viscous swirling flows. Our discussion centers on the implementation procedures based on spectral collocation technique. In viscous temporal analysis, the eigenvalue problem that govern the flow stability has been written in a matriceal form and a clustered grid has been set. In spatial analysis, the hydrodynamic model leads to a non-linear eigenvalue problem, that has been linearized using the companion vector method. Both methods provide an accurate approximation of the spectrum without any scale resolution restriction and relevant information on perturbation amplitudes for stable or unstable modes.
KEYWORDS:
spatial stability, temporal stability, swirling flow, collocation method.

1. INTRODUCTION

In most cases, the spatially or temporal stability (classified for open flows as in [8]) under infinitesimal perturbations is reduced to the study of an algebraic eigenvalue problem which imply to solve a dispersion relation, connecting the frequency ω and the axial wavenumber α as a consequence of the condition that nontrivial eigenfunctions exist. The instability of the flow is described by the dispersion relation in the spectral space (α,ω) corresponding to the spatio-temporal evolution of the most unstable mode in the physique space (r,t). Most of the investigations [6-12], concerned the values of the nondimenional parameters for which the vortex become unstable in the case of either a spatial stability or temporal stability analysis. When the complex frequency $\omega = \omega_r + i \cdot \omega_i$, $\omega_r = \mathrm{Re}(\omega)$, $\omega_i = \mathrm{Im}(\omega)$ is determined as a function of the real wave number k a temporal stability analysis is in fact performed. Conversely, solving the dispersion relation for the complex wave number, $\alpha = \alpha_r + i \cdot \alpha_i$, $\alpha_r = \mathrm{Re}(k)$, $\alpha_i = \mathrm{Im}(k)$, when ω is given real leads to the spatial branches $\alpha(\omega,\Psi)$ where by Ψ we denoted the set of all other physical parameters involved. In both cases, the sign of the imaginary part indicates the decay or either the growth of the disturbance.

Although a spatial stability analysis implies the investigation of a nonlinear eigenvalue problem, this type of analysis directly provides the frequency ranges of the most unstable modes. More than that, the spatial stability results can be directly compared to the experimental ones since usually, in experiments, an excitation is applied to a point in the flow and then, the effect of the excitation is studied as the flow evolves downstream.

This paper presents a numerical investigation upon the temporal and spatial stability of a swirl flow subject to infinitesimal perturbations. For the case of high Reynolds numbers the nonlinear eigenvalue problems governing the temporal and spatial stability analysis of the vortex structure is investigated using a spectral collocation technique. The numerical procedures need different approaches for temporal and spatial viscous studies, respectively. We developed numerical implementations which directly provides relevant graphic information about perturbation velocities amplitude for stable or unstable induced modes in both cases. The paper is outlined as follows: we set the problem and derive the mathematical models in Section 2. Section 3 describes in detail the numerical procedures based on collocation technique. Section 4 relates the numerical results and Section 5 concludes the paper.

2. VISCOUS HYDRODYNAMICAL MODELS FOR SPATIO-TEMPORAL STABILITY

The starting point to develop the mathematical model of a swirling fluid flow are usually the non-dimensionalized Navier-Stokes (NS) equations for incompressible flow, neglecting body forces and the continuity equation. The evolution of small perturbations of the basic flow is governed by the following dimensionless linearized NS equations, obtained after substituting the expressions for the components of the velocity and pressure field into the NS equations and only considering contributions of first order terms

$$\nabla\cdot \mathrm{v}=0\,,\ \frac{\partial \mathrm{v}}{\partial t}+(\mathrm{u}\cdot\nabla)\mathrm{v}+(\mathrm{v}\cdot\nabla)\mathrm{u}=-\nabla\pi+\frac{1}{\mathrm{Re}}\cdot\Delta\mathrm{v}\,, \tag{1}$$

where $\mathrm{u}=(u_r,u_\theta,u_z)$ represents the base velocity field and $\mathrm{v}=(\upsilon_r,\upsilon_\theta,\upsilon_z)$ is the small velocity perturbation.

The disturbance components of the velocity field and pressure into normal modes form are given here

$$\begin{aligned}&\upsilon_r=F(r)e^{i(\alpha z+n\theta-\omega t)},\quad \upsilon_\theta=iG(r)e^{i(\alpha z+n\theta-\omega t)},\\ &\upsilon_z=H(r)e^{i(\alpha z+n\theta-\omega t)},\quad \pi=P(r)e^{i(\alpha z+n\theta-\omega t)},\end{aligned} \tag{2}$$

where F,G,H,P represent the complex amplitudes of the perturbations, α is the axial wave number complex, n is the tangential integer wave number and ω represents the complex frequency. The factorization with respect to the axial coordinate z is allowed by the assumption on an axisymmetric parallel flow, the factorization in the azimuthal direction can be considered based on the axisymmetric flow assumption also. A linear stability study implies infnitesimal type perturbations so a factorization in time can be considered.

Assuming a steady columnar flow the velocity profile is written

$$\underline{V}(r)=[0,V(r),W(r)], \tag{3}$$

where V represents the tangential velocity component and W the axial component of the velocity, all depending only on radius.

The flow is assumed to be incompressible and the lengths in cylindrical coordinates are nondimensionalized with respect to the length scale $\sqrt{\nu/s}$, where ν is the viscosity of the fluid and s is the strain rate of the base flow field.

The hydrodynamic temporal stability model has the following form

$$\omega\mathrm{M}\bar{s}(r)\equiv-\mathrm{H}\bar{s}(r)\quad,\quad \bar{s}(r)=(F(r),G(r),H(r),P(r))^T\,. \tag{4}$$

Relation (4) consists in a set of partial differential equations for the perturbation velocities and pressure, expressed in a matriceal form. The non-zero elements of the 4 × 4 matrix operators M and H are given by

$$m_{11}=m_{22}=m_{33}=-i$$

and

$$h_{11}=\frac{Vin}{r}+Wi\alpha-\frac{1}{\mathrm{Re}}\left(\Delta+\frac{1}{r^2}\right),\ h_{12}=-\frac{2V}{r}+\frac{2in}{r^2\,\mathrm{Re}}\,,\ h_{14}=\frac{\partial}{\partial r}\,,$$

$$h_{21}=V'+\frac{V}{r}-\frac{2in}{r^2\,\mathrm{Re}}\,,\ h_{22}=\frac{Vin}{r}+Wi\alpha-\frac{1}{\mathrm{Re}}\left(\Delta-\frac{1}{r^2}\right),\ h_{24}=\frac{in}{r}\,,$$

$$h_{31}=W'\,,\ h_{33}=\frac{Vin}{r}+Wi\alpha-\frac{1}{\mathrm{Re}}\Delta\,,\ h_{34}=i\alpha\,,$$

$$h_{41}=\frac{\partial}{\partial r}+\frac{1}{r}\,,\ h_{42}=\frac{in}{r}\,,\ h_{43}=i\alpha\,,$$

where $\Delta\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-\alpha^2$ and primes denote $V'\equiv dV/dr$ and $W'\equiv dW/dr$. The Reynolds number Re is defined according to the maximum difference of axial velocity and the vortex core radius, as described in [2].

When we decompose the amplitude $F=F_r+iF_i$ and temporal frequency $\omega=\omega_r+i\omega_i$, one sees how the real and imaginary part of F and ω contributes to the wave solution

$$e^{\omega_i t}\{[F_r\cos(\omega_r t)+F_i\sin(\omega_r t)]\cos\Theta-[F_i\cos(\omega_r t)-F_r\sin(\omega_r t)]\sin\Theta\}$$
$$+ie^{\omega_i t}\{[F_r\cos(\omega_r t)+F_i\sin(\omega_r t)]\sin\Theta+[F_i\cos(\omega_r t)-F_r\sin(\omega_r t)]\cos\Theta\},\Theta\equiv n\theta+\alpha z.$$

The flow is considered unstable when the disturbance grows, i.e. the imaginary part of the eigenfrequency ω is positive.

In spatial stability case, the eigenvalue problem governing the hydrodynamical stability has the form

$$\left(\alpha^2\cdot L_{\alpha^2}+\alpha\cdot L_\alpha+L\right)\cdot\overline{S}=0\,, \tag{5}$$

where $\overline{S}=(F\quad G\quad H\quad P)^T$ and matrices L_{α^2}, L_α, L being of form

$$L_{\alpha^2}=\frac{1}{\mathrm{Re}}\begin{pmatrix}0&0&0&0\\-1&0&0&0\\0&-1&0&0\\0&0&-1&0\end{pmatrix},\quad L_\alpha=\begin{pmatrix}0&0&1&0\\W&0&0&0\\0&W&0&0\\0&0&W&1\end{pmatrix},\quad L=L_A+\frac{L_B}{\mathrm{Re}},$$

$$L_A=\begin{pmatrix}D+1/r & in/r & 0 & 0\\ i(-\omega+nV/r) & -2V/r & 0 & D\\ V'+V/r & i(-\omega+nV/r) & 0 & in/r\\ W' & 0 & i(-\omega+nV/r) & 0\end{pmatrix},$$

$$L_B=\begin{pmatrix}0&0&0&0\\ -D^2+(n^2+1)/r^2 & 2in/r^2 & 0 & 0\\ -2in/r^2 & -D^2+(n^2+1)/r^2 & 0 & 0\\ 0 & 0 & -D^2+n^2/r^2 & 0\end{pmatrix},$$

where $D\equiv\frac{d}{dr}$, $D^2\equiv\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$ and derivatives are calculated with respect to the radius.

The growth of the wave solution in spatial case depends on the imaginary part of the axial wavenumber, as described in the next formula

$$e^{-\alpha_i z}\{F_r\cos(\alpha_r z+\Theta)-F_i\sin(\alpha_r z+\Theta)+i[F_r\sin(\alpha_r z+\Theta)+F_i\cos(\alpha_r z+\Theta)]\},\ \Theta\equiv n\theta-\omega t.$$

Here the flow is considered unstable when the disturbance grows, i.e. the imaginary part of α is negative.

3. NUMERICAL ALGORITHMS BASED ON SPECTRAL COLLOCATION

The spectral collocation method is associated with a grid, that is a set of nodes and that is why it is sometimes referred to as a *nodal method*. The unknown coefficients in the approximation are then obtained by requiring the residual function to be zero exactly at a set of nodes. The set of the collocation nodes is related to the set of basis functions as the nodes of the quadrature formulae which are used in the computation of the spectral coefficients from the grid values.

Instead of representing the unknown function through its values on a finite number of grid points as doing in finite difference schemes, in spectral methods the coeficients $\{f_i\}_{i=0..N}$ are used in a finite basis of known functions $\{\Phi_i\}_{i=0..N}$

$$(F,G,H,P)=\sum_{i=0}^{N}(f_i,g_i,h_i,p_i)\,\Phi_i(r). \tag{6}$$

The decomposition (6) is approximate in the sense that $\{\Phi_i\}_{i=0..N}$ represent a complete basis of finite-dimensional functional space, whereas (F,G,H,P) usually belongs to some other infinite-dimensional space. Moreover, the coefficients (f_i,g_i,h_i,p_i) are computed with finite accuracy. Among the major advantages of using spectral methods is the rapid decay of the error, often exponential e^{-N} for well-behaved functions.

We choose for our study a Chebyshev Gauss-Lobatto collocation grid, that may be expressed as

$$\xi_j=\cos(\pi j/N),\ j=0..N. \tag{7}$$

This formula has the advantage that in floating-point arithmetic it yields nodes that are perfectly symmetric about the origin, being clustered near the boundaries diminishing the negative effects of the Runge phenomena[16]. This collocation nodes are the roots of Chebyshev polynomials and distribute the error evenly and exhibit rapid convergence rates with increasing numbers of terms.

When doing simulations and solving PDEs, a major problem is one of representing an deriving functions on a computer, which deals only with finite integers. In order to compute the radial and pressure derivatives that appear in our mathematical model, the derivatives are approximated by differentiating a global interpolant built trough the collocation points.

We choose $\{\Phi_i\}_{i=0..N}$ given by Lagrange's formula

$$\Phi_i(r)=\frac{\omega_N(r)}{\omega'_N(r_i)(r-r_i)}, \text{ where } \omega_{N(r)}=\prod_{m=1}^{N}(r-r_m). \tag{8}$$

We constructed an interpolative spectral differentiation matrix $D_{(N+1)\times(N+1)}$, having the entries

$$d_{00}=\frac{2N^2+1}{6},\ d_{NN}=-\frac{2N^2+1}{6},\ d_{jj}=\frac{-\xi_j}{2(1-\xi_j^2)},\ j=1,\ldots,N-1,$$

$$d_{ij}=\frac{c_i}{c_j}\frac{(-1)^{i+j}}{(\xi_i-\xi_j)},\ i\neq j,\ i,j=1,\ldots,N-1,\ c_i=\begin{cases}2 & \text{if } i=0,N\\ 1 & \text{otherwise}\end{cases}.$$

We made use of the conformal transformation, also used in [8]

$$r(\xi)=\frac{[1+b\exp(-a)]r_{\max}}{\left[1+b\exp\left(-a\frac{1-\xi}{2}\right)\right]}\left(\frac{1-\xi}{2}\right), \tag{9}$$

that maps the standard interval $\xi\in[-1,1]$ onto the phisical range of our problem $r\in[0,r_{\max}]$.

Because large matrices are involved, we numerically solved the eigenvalue problem using the Arnoldi type algorithm [5], which provides entire eigenvalue and eigenvector spectrum.

For non-axisymmetrical modes (cases having $|n|>1$), our boundary value problem have been solved subject to Dirichlet boundary conditions. This was numerically implemented as part of spectral collocation method by discarding the no effect first and last columns of the Chebyshev differentiation matrices of first and second order and also ignoring the first and last rows.

In spatial analysis, the nonlinear eigenvalue problem that govern the flow stability can be transformed into a linear problem, using the „companion vector method". We define the aditional vector $\bar{h}$ such that the eigenvalue problem can be written in form

$$\begin{bmatrix}L & 0\\ 0 & I\end{bmatrix}\begin{pmatrix}S\\ \bar{h}\end{pmatrix}+\alpha\begin{bmatrix}L_\alpha & L_{\alpha^2}\\ -I & 0\end{bmatrix}\begin{pmatrix}S\\ \bar{h}\end{pmatrix}=0,\quad \bar{h}=\begin{pmatrix}\alpha F\\ \alpha G\\ \alpha H\end{pmatrix}. \tag{10}$$

The mathematical models are completed in the both types of stability analysis with the boundary conditions from Batchelor [2] given in detail also in [8].

$$(|n|>1),\quad F=G=H=P=0, \tag{11a}$$

$$(n=0),\quad G=H=0, F, P \text{ finite}, \tag{11b}$$

$$(n=\pm1),\quad H\pm G=0, F=P=0. \tag{11c}$$

4. NUMERICAL RESULTS

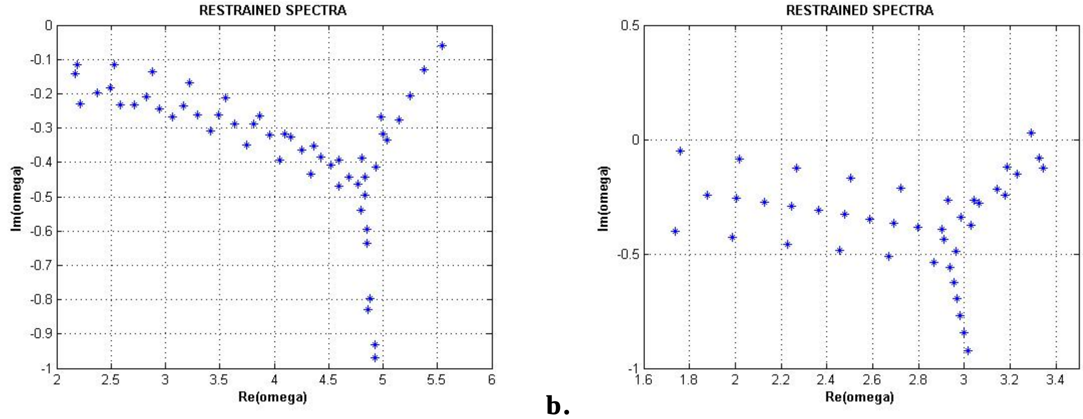
We consider a swirl flow [1] characterized by the velocity field

$$V(r)=\frac{q}{r}\left(1-e^{-r^2}\right),\ W(r)=a+e^{-r^2}, \tag{12}$$

where V represents the tangential velocity component and W the axial component of the velocity, q represents the swirl number defined as the angular momentum flux divided by the axial momentum flux and a provides a measure of free-stream axial velocity. In [13] a Chebyshev spectral collocation method for temporal and spatial stability is also presented.

In temporal stability analysis, in case n = 2 for given parameters Re = 8000, α = 3.5, q = 1, the fluid system is stable, as can be seen in Figure 1a.

For axysimmetrical mode n = 0, having the same initial parameters, Re = 8000, α = 3.5, q = 1, an eigenvalue with positive imaginary part occurred in spectra, thus the conclusion that the fluid system became unstable (Figure 1b).



a. **b.**

Figure 1. The „Y" shape of the eigenvalue spectra, for N=100 collocation points
a. Stable fluid system in case $n = 2$, $\mathrm{Re} = 8000$, $\alpha = 3.5$, $q = 1$;
b. Unstable fluid system in case $n = 0$, $\mathrm{Re} = 8000$, $\alpha = 3.5$, $q = 1$

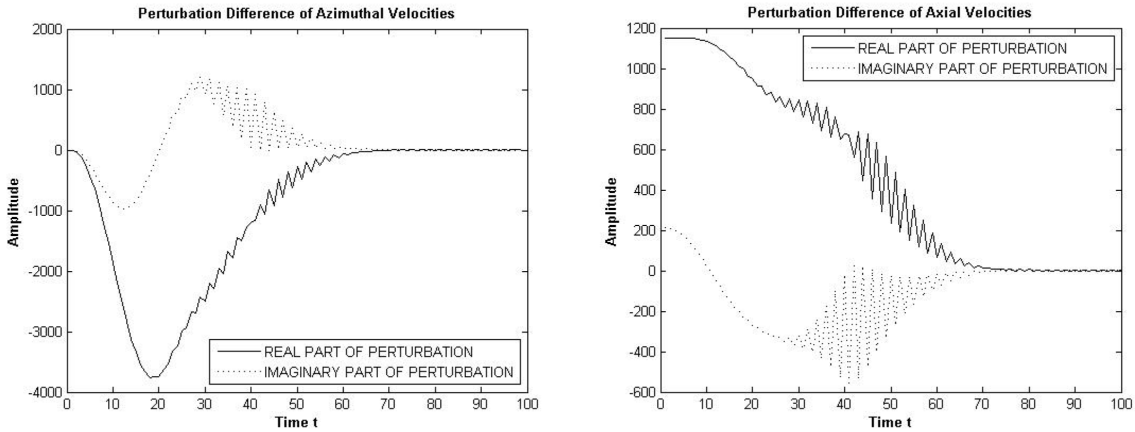


Figure 2. Variations in time of the differences between perturbed velocity fields developed for the first unstable eigenfrequency and the least stable one

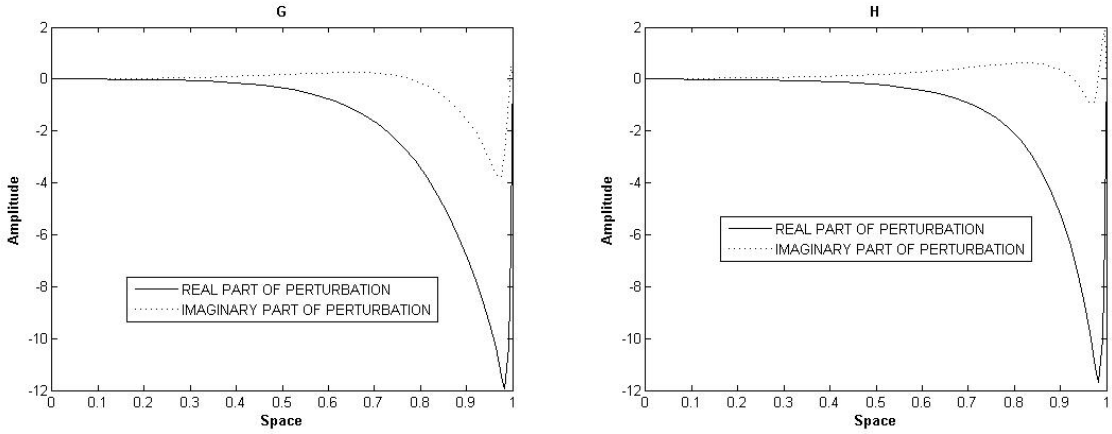


Figure 3. Spatial tangential perturbation G and axial perturbation H for non-axysymetrical mode $n = 3$, with parameters $N = 50$, $Re = 10000$, $\omega = -2.75$, $q = 0.1$.

We denote by *the least stable eigenfrequency*, the stable eigenvalue with maximum imaginary part, and by *the first unstable eigenfrequency*, the unstable eigenvalue having minimum imaginary part. The least stable eigenfrequency is $\omega_{LST} = 1.7591 - 0.051863i$, and the first unstable one $\omega_{FUS} = 3.2934 + 0.02789i$. Figure 2 presents the variations in time of the differences between perturbed velocity fields developed for temporall frequencies ω_{FUS} and ω_{LST}. Figure 3 shows the tangential perturbation and the axial perturbation for non-axysymetrical mode, computed in spatial stability analysis.

5. CONCLUSION

We have made a numerical investigation upon the spatio-temporall stability of a swirl flow, subject to infinitesimal perturbations. We have derived the hydrodynamic models describing the spatio-temporal stability and we have implemented numerical algorithms based on Chebyshev spectral collocation technique. We numerically obtained information on the state of the fluid system. For fixed parameters Re, α, and q, the stable system in non-axisymmetricall mode became unstable if the tangential wavenumber n is set to zero. We obtained a graphical visualization of the differences between perturbed velocities developed for the first unstable temporal frequency and the least stable one, in both the axisymmetrical and non- axisymmetrical modes.

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