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APPLICATION OF STABILITY THEORY OF NONLINEAR SYSTEMS AND LYAPUNOV TRANSFORMATION IN CONTROL OF ARTIFICIAL PNEUMATIC MUSCLE

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ABSTRACT: The following paper explores the concept of nonlinear systems stability and characteristic exponent. Definitions and theorems, necessary for solving the predefined problem of control of a nonlinear system, are included. The paper also deals with the Lyapunov transformation to carry out a linear system whose matrix elements are functions of a system with a constant matrix. The stability of systems with changeable parameters as well as the application of nonlinear systems control theory to the problems of "artificial pneumatic muscle - APM" control have also been investigated in the paper.
KEYWORDS: cybernetics, nonlinear systems stability, Lyapunov transformation, artificial pneumatic muscle

❖ INTRODUCTION

The stability of a given system is often defined in the sense that the system is capable of returning to an equilibrium if a signal acting, which led the system out of this state, finished. This definition is sufficient for a linear system, its stability, however, can be defined in a different way, e.g. a linear system is stable if and only if its response to an arbitrary bounded input is bounded.

There are several definitions of a nonlinear system stability. Many of them have a limited utilization and were defined for specific cases. In general, the processes going on in linear and nonlinear systems can be expressed by a mathematical model, which actually is a system of differential equations. Lyapunov stability theory enables to investigate the system stability without the necessity of solving either differential equations of the given order or a system of differential equations. A. M. Lyapunov proposed two methods in order to investigate the stability. Lyapunov first method enables to consider the nonlinear system stability according to an approximate linear model, (local stability). Lyapunov second method enables to consider the stability or the asymptotic stability in a certain area Ω, in general with the linear or nonlinear system, (of both excited and unexcited system). When solving the stability problem, the success of the method lies with the investigator'stability to find a suitable function (the so called Lyapunov function) as well as to determine its definiteness, [1, 3, 4, 5].

This paper will deal with the investigation of nonlinear systems stability described by a vector differential equation, a characteristic exponent and an asymptotic stability. It will also deal with the Lyapunov transformation as well as the stability of the systems with variable coefficients of the system of differential equations.

❖ SYSTEM STABILITY AND CHARACTERISTIC EXPONENT

We will consider a homogeneous linear vector differential equation (or a homogeneous linear system of differential equations) in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{1}$$

where matrix individual elements

$$\mathbf{A}(t) = (a_{ij}(t)) \tag{2}$$

are continuous functions in the interval $(a, +\infty)$.

Theorem 1. A linear system described by the equation (1) is stable in the sense of Lyapunov in the interval $< t_0, +\infty >$, if all solutions to the equation (1) are bounded functions in the interval $< t_0, +\infty >$.

Theorem 2 (R. Bellman). Let all the solutions to the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}.\mathbf{x}(t) \tag{3}$$

with a constant matrix of (n , n) type be stable in the sense of Lyapunov, or let all the roots of the equation det **A** = 0 have negative real parts. Let **B**(t) be a matrix, whose elements are variables of the (n, n) type, where its elements are continuous functions in the interval $< t_0, +\infty >$ and let the integral be expressed in the form

$$\int_{t_0}^{\infty} \|\mathbf{B}(\tau)\|.d\tau < 0, \tag{4}$$

then all the solutions to the equation

$$\dot{\mathbf{y}}(t) = [\mathbf{A} + \mathbf{B}(t)].\mathbf{y}(t) \tag{5}$$

are stable in the interval $< t_0, +\infty >$ in the sense of Lyapunov.

❖ CHARACTERISTIC EXPONENT

First, we will present stability conditions of linear systems with variable coefficients. The basic notion is that of a characteristic exponent of the function introduced by A.M. Lyapunov.

Definition 1. A characteristic exponent of a complex function f(t) of a real variable t is the number:

$$\chi(t) = \lim_{t \to \infty} \sup (t)^{-1}.\ln|f(t)| \tag{6}$$

In order to explain this notion by which a growth velocity of a function is characterized, it is sufficient to realize the following fact. A module of the given function can be expressed in the form

$$|f(t)| = e^{\alpha(t).t} \tag{7}$$

whereas

$$\alpha(t) = (t)^{-1}.\ln|f(t)| \tag{8}$$

Asymptotic behaviour of this function α(t) for $t \to \infty$ is expressed by the relation (6). Obviously for the real α it holds $\chi(e^{\alpha t}) = \alpha$

It is possible to derive many basic properties of a characteristic exponent, [3].

Theorem 3. (A. M. Ljapunov). If the matrix (2) in the equation (1) is norm - bounded (we can assume an arbitrary norm in the relation (11)).

$$\|\mathbf{A}(t)\| \le C < +\infty , \tag{9}$$

then every non-zero solution **x**(t) has the infinite characteristic exponent.

Definition 2. A set of all finite characteristic exponents of the solutions to the system of differential equations (nonlinear in general) is called a spectrum of this system.

First, let us assume the system of first order linear differential equations with a matrix, whose elements are constants (1). In general, each component of the solution to this system can be expressed in the form of a linear combination

$$x_j(t) = \sum_{i=1}^{m} c_{ij} P_i(t) e^{\lambda_i t}$$

where $P_i(t)$ are the polynomials in t and λ_i are eigennumbers of the matrix **A**, or the roots of the equation

$$\det(\mathbf{A} - \lambda E) = 0 \quad \text{thus it is}$$

$$\chi(P(t)e^{\lambda t}) = R_e \lambda$$

and a characteristic exponent of the solution ***x(t)*** is thus equal to a real part of some of the matrix **A** eigenvalues. In the sense of the definition 2, spectrum is identical to the sets of real parts of the matrix **A** eigenvalues.

The case in which the **A** matrix in (2) has changeable elements will be described in the following theorem.

Theorem 4. Spectrum of the system of linear homogenous differential equations (1) of the m - th order is a finite set of numbers

$$\alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_m \quad , \text{where } m \le n \tag{10}$$

❖ ASYMPTOTIC STABILITY OF A VECTOR DIFFERENTIAL EQUATION AND A CHARACTERISTIC EXPONENT

Theorem 5. In the sense of Lyapunov, for the asymptotic stability of a linear homogeneous system described by the vector equation (1) it is sufficient that its maximum characteristic exponent is negative.

Proof. Let the maximum element of the **A**(t) matrix spectrum according to the definition 3 be α < 0. Due to the fact that it refers to a linear system, it is sufficient to prove that for each solution it holds:

$$\lim_{t\to+\infty}\|\mathbf{x}(t)\| = 0 \tag{11}$$

Let us consider such number β, that $\alpha<\beta<0$, then based on the inequality

$$\chi(\mathbf{x})<\beta \quad \text{it follows}$$

$$\|\mathbf{x}(t)\| = \mathbf{O}(e^{\beta t}),$$

thus, it really holds (11).

Using the characteristic exponents, it is possible to characterize a set of solutions to the linear homogeneous system as follows, let

$$\{\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots\ldots, \mathbf{x}_n(t)\} \tag{12}$$

be a fundamental system of solutions to the equation (1) and let

$$\chi(\mathbf{x}_j) = \alpha_j, \quad j = 1,2,\ldots\ldots,n \tag{13}$$

We remark that the numbers (28) are not necessarily mutually different.

Let us denote

$$\mathbf{x}_j(t) = e^{\alpha_j(t)}.\xi_j(t) \tag{14}$$

From the above, it follows that

$$\chi(\xi_j) = 0 \tag{15}$$

In general, the solution to the system (1) has the form:

$$x(t) = \sum_{j=1}^{n} c_j \xi_j(t) e^{\alpha_j t} \tag{16}$$

where c_j are constants, $\xi_j(t)e^{\alpha_j t}$ are linear independent solutions (12), α_j are the elements of the matrix (2) spectrum and $\xi_j(t)$ have the property (15).

Let all the numbers (13) be finite and let the n - tuple (13) be created by mutually different numbers α_k , k = 1,2, ...,m, where, certainly, m ≤ n. Let v_k be a number of solutions (12), which have a characteristic exponent α_k. In dependence on a selected fundamental matrix (12), let us denote it $\boldsymbol{X}$, we can construct a number

$$s(X) = \sum_{k=1}^{m} v_k \alpha_k \tag{17}$$

Fundamental systems with a minimum number $s(\boldsymbol{X})$ are sometimes called normal. According to Lyapunov, it is possible to derive a low estimation of a number (17),[1,2,3].

❖ LYAPUNOV TRANSFORMATION

When investigating the stability of solutions to homogeneous linear systems (1) in some cases it is possible to find a linear transformation

$$\mathbf{y}(t) = \mathbf{L}(t)\,\mathbf{x}(t) \tag{18}$$

which will change a system (1) with the **A**(t) matrix to the system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{B}\mathbf{y}(t) \tag{19}$$

with a constant matrix. If during this transformation the characteristic exponents are not changed, it is possible to solve the problem of stability of the system (1) by means of known methods.

Definiton 3. The **L**(t) matrix, whose elements have continuous first derivatives in the interval $<t_0, +\infty)$, is called Lyapunov matrix, if

a) $\sup_{t\in<t_0,+\infty)}\|\mathbf{L}(t)\|$ and $\sup_{t\in<t_0,+\infty)}\left\|\frac{d\mathbf{L}(t)}{dt}\right\|$ are finite numbers

b) $\|\det \mathbf{L}(t)\| \ge k > 0 \quad \forall t \in <t_0, +\infty)$

The corresponding transformation (18) is called Lyapunov transformation.

❖ STABILITY OF SYSTEMS WITH VARIABLE COEFFICIENTS - BASIC RELATIONS

Differential equations which are used to describe the systems with variable parameters have time-varying coefficients; they will be denoted as $a_i(t)$ time functions. The stability of the systems with variable parameters can be secured only in a certain time interval. Beyond this interval, the system can be instable.

We will investigate the system with variable parameters described by the differential equation

$$a_n(t)x_2^{(n)}(t)+.....+a_1(t)x_2'(t)+a_0(t)x_2(t)=x_0(t) \tag{20}$$

$x_2(t)$ is an output value, $x_0(t)$ is an input value. Our task is to find a relation between the input and the output values of the investigated system for such a case that the system is in an equilibrium until the moment when the input signal starts acting,. The solution is considered from the moment when the input signal is applied. For this moment it holds:

$$x_2^{(v)}(t)|_{t=0}=0, \quad v=1,2,....,(n-1) \tag{21}$$

The solution to the equation (20) will be obtained by the variation of constants method, [3]. The considered solution is searched for in the form

$$x_2(t)=\varphi_1(t)y_1(t)+\varphi_2(t)y_2(t)+...+\varphi_n(t)y_n(t) \tag{22}$$

where $\varphi_i(t)$ are linear independent particular solutions to the homogeneous equation, $y_j(t)$ will be determined in such a way that after inserting the expression (22) into (20) we obtain the identity.

Thus we obtain the solution in the form:

$$x_2(t)=\int_{-\infty}^{\infty} g(t,u)x_0(u)du \tag{23}$$

In order to explain the physical substance of the g (t, u) function, we will investigate the case in which at the moment t = ξ for the system input there is introduced a signal in the form of the Dirac impulse, i. e.

$$x_0(u)=\delta(u-\xi), \quad 0<\xi<u$$

If we apply the expression (23) and the known equality

$$\int_{-\infty}^{\infty} f(t).\delta(t-\xi)dt=f(\xi)$$

thus we obtain an impulse transition function of the system, which is described by the equation (22):

$$\int_0^t g(t,u).\delta(t-\xi)du=g(t,\xi)$$

The impulse transition function is called the system response (which before the beginning of the signal acting was in equilibrium) to the input signal in the form of the Dirac impulse. Considering a mathematical point of view, $g(t,\xi)$ is the solution to the differential equation

$$a_n(t)g^{(v)}(t,\xi)+.....+a_1(t)g'(t,\xi)+a_0(t)g(t\xi)=\delta(t-\xi) \tag{24}$$

with the initial conditions $g^{(v)}(t,\xi)|_{t=\xi}=0, \quad v=0,1,2,...,(n-1)$

The impulse transition function can also be applied to a more general case for the systems with changeable parameters to solve the equation in the form:

$$\begin{aligned} &a_n(t)w^{(n)}(t,\xi)+.....+a_1(t)w'(t,\xi)+a_0(t)w(t\xi)= \\ &=b_m(t)\delta_m^{(m)}(t-\xi)+.....+b_1(t)\delta_1'(t-\xi)+b_0(t)\delta_0(t-\xi) \end{aligned} \tag{25}$$

with the initial conditions

$$w^{(v)}(t,\xi)|_{t=\xi}=0, \quad v=0,1,2,...,(n-1) \tag{26}$$

In this case, $w(t,\xi)$ represents an impulse transition function of the system with the changeable parameters of a general form. The $w(t,\xi)$ function is related [7, 8, 10, 11] to the $g(t,\xi)$ function according to the relation

$$w(t,\xi)=(-1)^m\frac{d^m}{d\xi^m}\left[g(t,\xi)b_m(\xi)\right]+...+g(t,\xi)b_0(\xi) \tag{27}$$

❖ CONTROL OF THE APM NON-LINEAR SYSTEM

From the theoretical point of view, modelling and control of a pneumatic actuator, called “artificial pneumatic muscle” (APM), is a complex problem. The APM control is considerably complicated owing to its simple design, especially because of its nonlinearity, air compressibility, time varying properties as well as the difficulties in the analytical modelling, fig 1a, b.

In general, APM is investigated from the viewpoint of the theory of nonlinear systems, since the mathematical model is expressed by a second order nonlinear differential equation in the form

$$M\ddot{\mathbf{x}}+B(\dot{\mathbf{x}})\dot{\mathbf{x}}+K(\mathbf{x})\mathbf{x}=u \tag{28}$$

or, assuming that a total mass is unity, i.e. $M=1$; and the remaining physical values have a usual meaning, [3, 4, 5, 8, 9], whereas the nonlinear functions are denoted $f_1(\dot{\mathbf{x}}), f_2(\mathbf{x})$, then the equation (28) can be expressed as follows:

$$\ddot{\mathbf{x}} + f_1(\dot{\mathbf{x}})\dot{\mathbf{x}} + f_2(\mathbf{x})\cdot\mathbf{x} = u^* \tag{29}$$

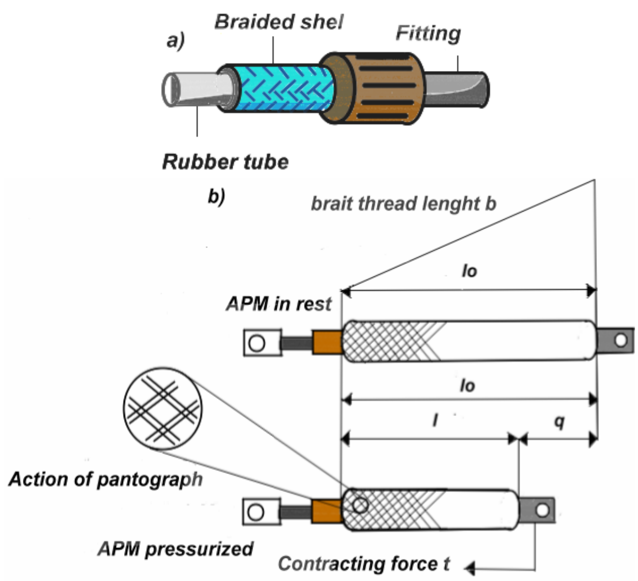

Fig. 1 - The basic design of the artificial pneumatic muscle (a), its functions (b)

This equation is equivalent to the system of the first order differential equations

$$(x_1 = x \quad a \quad x_2 = \dot{x}) \qquad \begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u - f_1(\dot{x})\dot{x} - f_2(x)x\end{aligned}$$

or in the matrix expression

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -f_2(x_1) & -f_1(x_2)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u$$

and

$$y(t) = x(t) = x_1(t)$$

Further on, we assume that nonlinear functions $f_1(x_2), f_2(x_1)$ can be expressed by means of second order polynomials, i.e.

$$f_1(x_2) = B(x_2)\,x_2$$
$$f_2(x_1) = K(x_1)\,x_1$$

and let us write

$$B(x_2) = b_2x_2^2 + b_1x_2 + b_0$$
$$K(x_1) = a_2x_1^2 + a_1x_1 + a_0$$

The $b_i, a_i\ (i = 0,1,2)$ coefficients can be obtained by the identification of the measured values

$K(x_1) = K(x)$ represents a model (inflation and deflation - hard spring)

$B(x_2) = B(\dot{x})$ represents a model (inflation and deflation), [3, 4, 10].

❖ LYAPUNOV FUNCTION AND LAW OF CONTROL

If there is a mathematical model (APM) designed, which is represented by a second order nonlinear differential equation, it is possible to express a Lyapunov function and a control:

$$M\ddot{x} + f_1(\dot{x})\dot{x} + f_2(x)x = u \equiv F \tag{35}$$

$$V_2 = \frac{1}{2}M\dot{x}^2 + \int_0^x f_2(\tau)\tau\, d\tau \tag{36}$$

$$\dot{V}_2 = F\dot{x} - \left[f_1(\dot{x})\,\dot{x}^2\right] \tag{37}$$

Let us denote the $x_d(t)$ reference trajectory and let u(t) be clearly known, then the relations (14), (15) are valid. For a feedback after a linearization, it is possible to write dependence:

$$u = f_2(x_1)x_1 + f_1(x_2)x_2 + \ddot{x}_d - \lambda\dot{x} \tag{38}$$

Then the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \ddot{x}_d - \lambda\left|\dot{x}_1 - \dot{x}_d\right|\end{aligned} \tag{39}$$

is stable. Based on this it follows that

$$\dot{x}_2 + \lambda x_2 = \ddot{x}_d + \lambda\dot{x}_d \tag{40}$$

By virtue of Lyapunov theory, it is possible to show that the derivatives of the reference trajectory $x_d(t)$ are of the exponential order, thus the solutions $x_1(t)$ and $x_2(t)$ are exponentially stable, [4, 5, 8, 10].

❖ CONCLUSIONS

The contribution of the paper consists of the processing of the achieved results based on a considerably wide theoretical part on stability of linear and nonlinear systems expressed by a mathematical model which is represented by a homogeneous linear system of differential equations (a homogeneous linear vector differential equation) with changeable coefficients based on the first and second Lyapunov method. In order to solve the problems of stability defined by a linear vector differential equation with the **A**(t) time-variant matrix, Bellman, Gronwald and Lyapunov lemmas and theorems were applied. This refers to the theorems utilizing the defined notion of a characteristic exponent, a matrix spectrum, but especially the Lyapunov transformation.

The essence of the presented methods of solution applied to the problems of asymptotic stability of the system with a time varying matrix lies in the application to the problem of the system stability with a constant matrix. The mentioned possibility of the problem solving is proved by means of the Lyapunov transformation and Levinson theorem. Another contribution lies in the solution of the problems of stability of the systems with time-variant parameters, which are described by a system of differential equations with time-varying coefficients. The paper presents the solution to the problem with the utilization of the variation of constants method and the notion of impulse transition function of the system. The presented theoretical knowledge is applied to the control of the APM nonlinear system.

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