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STABILITY RESULTS FOR SOME FIXED POINT ITERATIVE PROCESSES IN CONVEX METRIC SPACES

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ABSTRACT: In this paper, we obtain some stability results in complete convex metric spaces for selfmappings satisfying certain general contractivity conditions. Our results extend several stability results in the literature.
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INTRODUCTION

The notion of convexity in metric spaces was introduced by Takahashi [27] and he established that all normed spaces and their convex subsets are convex metric spaces. Inaddition, he gave several examples of convex metric spaces which are not imbedded in any normed space or Banach space. Several papers have been devoted to the study of convex metric spaces in the literature (see Agarwal et al [1], Beg [2, 3], Ciric [9], Guay et al [11] and Shimizu and Takahashi [26]).

Definition 1.1 [3, 27]: Let (X,d) be a metric space. A mapping $T: X \times X \times [0,1] \to X$ is said to be a *convex structure on X* if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda) d(u,y). \qquad (1)$$

A metric space X having the convex structure W is called a convex metric space.

Let (X,d,W) be a convex metric space. A nonempty subset E of (X,d,W) is said to be *convex* if $W(x,y,\lambda) \in E$ whenever $(x,y,\lambda) \in E \times E \times [0,1]$.

Takahashi [20] has also shown that the open ball $B(x,r) = \{y \in X \mid d(x,y) < r\}$ and the closed ball $\overline{B(x,r)} = \{y \in X \mid d(x,y) \le r\}$ are convex. See also [1, 3].

In a complete metric space setting, Harder and Hicks [8] defined stability of iterative procedures as follows:

Definition 1.2 [12]: Let (X,d) be a complete metric space and $T: X \to X$ a selfmapping. Suppose that $F_T = \{p \in X \mid Tp = p\}$ is the set of fixed points of T.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving T which is defined by

$$x_{n+1} = f(T, x_n), n = 0,1,2,\dots, \qquad (2)$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(y_{n+1}, f(T,y_n)), n = 0,1,2,\dots$ Then, the iteration procedure (2) is said to be T-stable or stable with respect to T if and only if $\lim_{x\to\infty} \varepsilon_n = 0$ implies $\lim_{x\to\infty} y_n = p$.

Several stability results have been obtained by various authors using different contractive definitions. Harder and Hicks [12] obtained interesting stability results for some iteration procedures using various contractive definitions. Rhoades [24, 25] generalized the results of Harder and Hicks [12] to a more general contractive mapping. Osilike and Udomene [20] generalized some of the results of

[12] and [25] by employing the following contractive definition: For $T: X \to X$, there exist a constant $L \geq 0$ and $a \in [0,1)$ such $\forall x, y \in X$,

$$\mathrm{d}(T\mathrm{x}, T\mathrm{y}) \leq L\mathrm{d}(\mathrm{x}, T x) + a\mathrm{d}(\mathrm{x}, \mathrm{y}). \tag{3}$$

Condition (3) is more general than those of [12, 25]. As in [12], Berinde [4] obtained the same stability results for the same iteration procedures using the same contractive definitions, but applied a different method. The method of Berinde [4] is similar to that employed by Osilike and Udomene [20]. If in (3), $L = 0$, then we obtain

$$\mathrm{d}(T\mathrm{x}, T\mathrm{y}) \leq a\mathrm{d}(\mathrm{x}, \mathrm{y}), \ \forall x, y \in X. \tag{4}$$

Condition (4) was one of the contractive conditions used by the author [12] to prove some stability results. Also, condition (3) reduces to the Zamfirscu contraction condition

$$\mathrm{d}(T\mathrm{x}, T\mathrm{y}) \leq \delta\mathrm{d}(\mathrm{x}, \mathrm{y}) + 2\delta\mathrm{d}(\mathrm{x}, T x), \ \ \forall x, y \in X, \tag{5}$$

if $L = 2\delta, a = \delta,$ where δ =max $\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}, 0 \leq \delta < 1.$

Zamfirescu [28] established a nice generalization of the Banach's fixed point theorem. More recently, Berinde [8] established several generalizations of Banach's fixed point theorem. In one of the results of [8], the following contractive condition was employed: For a mapping $T: X \to X$, there exists $\alpha \in$ [0; 1) and some $L \geq$ 0 such that for all x; y$\in X$, we have

$$d(Tx, Ty) \leq \alpha M(x, y) + Lm(x, y) \tag{6}$$

where $M(x, y)$ =max $\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$,

and $m(x, y) = \min\{$d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)}.

Many authors including those mentioned earlier in this paper have proved various stability results for the following iterative processes in normed spaces:

For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ defined iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \alpha_n \in [0,1], \tag{7}$$

is called the Mann iterative process (see [17]).

For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, n = 0,1,2,... \end{aligned} \tag{8}$$

where $\alpha_n, \beta_n \in [0,1]$, is called the Ishikawa iterative process (see [14]).

It has been shown in many papers that both Mann and Ishikawa iterative processes reduce to the Picard iteration

$$x_{n+1} = T x_n, n = 0,1,2,..., \tag{9}$$

for which also stability results have been established in metric spaces for certain contractive mappings by most authors.

Lemma 1.1 [4, 7]: If δ is a real number such that $0 \leq \delta$ < 1, and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{x\to\infty} \varepsilon_n = 0,$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \varepsilon_n$, n = 0; 1;2,... ; we have $\lim_{x\to\infty} u_n = 0.$

❖ PRELIMINARIES

In this section, we introduce a definition of $T-$stability in convex metric space setting:

Definition 2.1: Let (X, d, W) be a convex metric space and $T: X \to X$ a selfmapping. Suppose that $F_T = \{p \in X \mid Tp = p\}$ is the set of fixed points of T.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative procedure involving T which is defined by

$$x_{n+1} = f_{T,\ \alpha_n}^{x_n}, \ \ n = 0, 1, 2, \ldots, \tag{10}$$

where $x_0 \in X$ is the initial approximation and $f^{x_n}_{T,\alpha_n}$ is some function having convex structure such that $\alpha_n \in [0,1]$. Suppose thatt $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set $\varepsilon_n = d(y_{n+1}, f^{y_n}_{T,\alpha_n})$, $(n = 0,1,2,\ldots)$. Then, the iterative procedure (10) is said to be T-stable or stable with respect to T if and only if $\lim_{x\to\infty} \varepsilon_n = 0$. implies $\lim_{x\to\infty} y_n = p$.

Remark 2.1: In terms of convex structure, we have the following:

(i) If in (10), $f^{x_n}_{T,\alpha_n} = W(x_n, Tx_n, \alpha_n)$, $\alpha_n \in [0,1]$, then we obtain Mann iteration

$$x_{n+1} = W(x_n, Tx_n, \alpha_n),\ \alpha_n \in [0,1]. \qquad (\lhd)$$

(ii) From (10) again, if $f^{x_n}_{T,\alpha_n} = W(x_n, Tb_n, \alpha_n)$, we obtain Ishikawa iterative process

$$x_{n+1} = W(x_n, Tb_n, \alpha_n), \text{ with } b_n = W(x_n, Tx_n, \beta_n),\ \alpha_n, \beta_n \in [0,1]. \qquad (\lhd\rhd)$$

The iterative schemes in ($\lhd$) and ($\lhd\rhd$) are the equivalent forms of (7) and (8) respectively in the convex metric space setting.

(iii) In a similar manner to the above, several other iterative processes can be deduced from (10).

Motivated by the work of Berinde [8], we now state the following contractive condition which shall be employed in the sequel:

For a mapping $T : X \to X$, there exists $\delta \in [0,1)$ and $L \geq 0$ such that $\forall x, y \in X$, we have

$$d(Tx, Ty) \leq \delta d(x, y) + Lm(x, y)\,, \qquad (11)$$

where

$$m(x,y) = \min\left\{ d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), \frac{1}{2}[d(x,Tx)+d(y,Ty)], \frac{1}{2}[d(x,Ty)+d(y,Tx)] \right\}.$$

Remark 2.2: It has been shown in paper [19] that the contractive condition (11) is more general than a number of contractivity conditions in the literature. However, condition (11) is independent of condition (6) (see Berinde [8]).

❖ MAIN RESULTS

Theorem 3.1: Let (X, d, W) be a complete convex metric space and $T : X \to X$ a mapping satisfying contractive condition (11). Suppose that T has a fixed point p. For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty}$ defined by ($\lhd$) be the Mann iterative process, where $\alpha_n \in [0,1]$ such that $0 < \alpha \leq \alpha_n$. Then, the Mann iteration is T - stable.

Proof: Let $\{y_n\}_{n=0}^{\infty} \subset X$ and define $\varepsilon_n = d(y_{n+1}, W(y_n, Ty_n, \alpha_n))$. Suppose that $\lim_{x\to\infty} \varepsilon_n = 0$.

Then, we establish that $\lim_{x\to\infty} y_n = p$. by using condition (11). Thus, we have that

$$\begin{aligned}
d(y_{n+1}, p) &\leq d(y_{n+1}, W(y_n, Ty_n, \alpha_n)) + d(W(y_n, Ty_n, \alpha_n), p)\\
&= d(W(y_n, Ty_n, \alpha_n), p) + \varepsilon_n = (1-\alpha_n) d(y_n, p) + \alpha_n d(Ty_n, p)\\
&= (1-\alpha_n) d(y_n, p) + \alpha_n d(Ty_n, Tp) + \varepsilon_n\\
&\leq (1-\alpha_n) d(y_n, p) + \alpha_n[\delta d(y_n, p) + Lm(y_n, p)] + \varepsilon_n\\
&= (1-\alpha_n) d(y_n, p) + \delta\alpha_n d(y_n, p) + \varepsilon_n \ \left(\sin ce\ m(y_n, p) = 0\right)\\
&= [1-(1-\delta)\alpha_n]\, d(y_n, p) + \varepsilon_n \leq [1-(1-\delta)\alpha] d(y_n, p) + \varepsilon_n
\end{aligned} \qquad (12)$$

Therefore, since $0 \leq 1-(1-\delta)\alpha < 1$, applying Lemma 1.1 in (12) yields

$$\lim_{x\to\infty} d(y_n, p) = 0, \text{ that is, } \lim_{x\to\infty} y_n = p.$$

Conversely, let $\lim_{x\to\infty} y_n = p$. Then, we prove that $\lim_{x\to\infty} \varepsilon_n = 0$. Thus, we have

$$\begin{aligned}
\varepsilon_{n0} &= d(y_{n+1}, W(y_n, Ty_n, \alpha_n)) \leq d(y_{n+1}, p) + d(p, W(y_n, Ty_n, \alpha_n))\\
&\leq d(y_{n+1}, p) + (1-\alpha_n) d(y_n, p) + \alpha_n d(Tp, Ty_n)\\
&\leq d(y_{n+1}, p) + [1-(1-\delta)\alpha] d(y_n, p) \to 0 \quad as \quad n \to \infty.
\end{aligned}$$

Theorem 3.2: Let (X, d, W) be a complete convex metric space and $T : X \to X$ a mapping satisfying contractive condition (11). Suppose that T has a fixed point p. For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty}$

defined by (◁▷) be the Ishikawa iterative process, where $\alpha_n, \beta_n \in [0,1]$ such that $0 < \alpha \le \alpha_n$, $0 < \beta \le \beta_{n.}$. Then, the Ishikawa iteration is T - stable.

Proof: Let $\{y_n\}_{n=0}^{\infty} \subset X$ and define $b_n = W(y_n, Ty_n, \beta_n)$, $\varepsilon_n = d(y_{n+1}, W(y_n, Tb_n, \alpha_n))$. Suppose that $\lim_{x\to\infty} \varepsilon_n = 0$. Then, we establish that $\lim_{x\to\infty} y_n = p$. by using condition (11). Thus, we have that

$$d(y_{n+1}, p) \le d(y_{n+1}, W(y_n, Tb_n, \alpha_n)) + d(W(y_n, Tb_n, \alpha_n), p) \tag{13}$$

$$\le (1-\alpha_n) d(y_n, p) + \alpha_n d(Tp, Tb_n) + \varepsilon_n \qquad \le (1-\alpha_n) d(y_n, p) + \delta\alpha_n d(p, b_n) + \varepsilon_n.$$

Also, $d(p, b_n) = d(p, W(y_n, Ty_n, \beta_n)) \le (1-\beta_n) d(y_n, p) + \beta_n d(Tp, Ty_n)$

$$\le (1-\beta_n) d(y_n, p) + \delta\beta_n d(y_n, p)) \tag{14}$$

Using (14) in (13) gives $d(y_{n+1}, p) \le [1-(1-\delta)\alpha_n - (1-\delta)\delta\alpha\beta_n] d(y_n, p) + \varepsilon_n$

$$\le [1-(1-\delta)\alpha - (1-\delta)\delta\alpha\beta\,] d(y_n, p) + \varepsilon_n\,. \tag{15}$$

Since $0 \le 1-(1-\delta)\alpha - (1-\delta)\delta\alpha\beta < 1$, using Lemma 1.1 in (15) yields $\lim_{x\to\infty} d(y_n, p) = 0$, that is, $\lim_{x\to\infty} y_n = p$. Conversely, let $\lim_{x\to\infty} y_n = p$. Then, $\varepsilon_n = d(y_{n+1}, W(y_n, Tb_n, \alpha_n))$

$$\le d(y_{n+1}, p) + [1-(1-\delta)\alpha - (1-\delta)\delta\alpha\beta\,] d(y_n, p) \to 0 \text{ as } n \to \infty.$$

Remark 3.1: Both Theorem 2.1 and Theorem 2.2 extend and generalize analogous results from [4, 7, 12, 20,24, 25]. See also some other results of the author [18]

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