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WAVE PROPAGATION IN PLATES OF ANISOTROPIC MEDIA ON THE BASIS EXACT THEORY

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ABSTRACT: The wave propagation in plates of general anisotropic media is investigated on the basis of an exact theory. Initially a formal analysis of waves in anisotropic elastic plate with 21 independent elastic constants is considered. Calculation is then carried forward for more specialized cases higher symmetry materials such as monoclinic to isotropic. Invoking the suitable boundary conditions on the plate outer surfaces, variety of important problems can be solved. Of these mentions, are stress-free boundaries and constrained boundary conditions. It is also exhibit that the particle motions for SH modes decouple if the propagation occurs along an in-plane axis of symmetry. Solutions that are bound near the surfaces in the limit $d \to \infty$ are also considered. Some special cases have also been deduced and discussed. The results obtained theoretically have been verified numerically and illustrated graphically.
KEYWORDS: Anisotropic, stress-free, symmetric modes, antisymmetric modes

❖ INTRODUCTION

The growing use of engineering materials such as fiber reinforced composite, graphite and laminate in recent years has resulted in considerable research activities on the behavior of such materials. Composite materials are ideal for structural applications where high strength-to-weight and stiffness-to-weight ratios are required. Hence, the studies of wave propagation in such materials which are anisotropic in nature become very important.

Propagation of guided elastic waves in plate structures has attracted a great deal of research interest since the pioneering work of Lord Rayleigh in 1889 [1]. Based upon Rayleigh's mathematical formulation, Lamb [2] extensively investigated the properties of these waves and today they are known as Rayleigh-Lamb waves. However, the mathematical complexity of the formulation as well as the fact that these waves can only be propagated at ultrasonic frequencies resulted in little advancement in knowledge of plate wave characteristics until recently. Through the work of modern investigators [3-7], the precise nature of the Rayleigh-Lamb spectrum was fully revealed. While the bulk of research work in this area has been devoted to isotropic media, there has been limited attention paid to anisotropic materials. The mathematical basis for plate wave propagation in anisotropic media was first presented by Ekstein [8] in 1945. Mindlin motivated by an interest in the vibration of quartz piezoelectric transducers, used the Ekstein formulation to examine plate waves in crystals of monoclinic symmetry along symmetry.

Compared to the extensive literature on the elastic waves in infinite anisotropic media; relatively little attention has been given to elastic waves in anisotropic plates. Although a complete review of the extensive literature on this subject cannot be undertaken, several salient contributions should be mentioned. Propagation of elastic waves in anisotropic homogeneous plate has been studied in detail by authors [9-15]. These studies provide an interesting picture of the rich dispersion characteristic of these guided waves. Several others authors [16-18] have studied free Lamb waves. Wave propagation in anisotropic composite media has also been studied by Nayfeh and Chementi [19] and Yan Li & R. B Thomson [20].

In this paper the propagation in plates of general anisotropic media is investigated on the basis of an exact theory. The wave is allowed to propagate along an arbitrary angle from the normal to the plate as well as any azimuthal angle. The secular equation for elastic waves is determined by expressing the stresses and displacements on one side of the plate to those of the other side in the matrix form. The calculations are then carried forward for more specialized case of a monoclinic plate. The corresponding results for materials systems of higher symmetry, such as orthotropic, transversely isotropic, cubic, and isotropic are contained implicitly in our analysis. We also demonstrate that the particle motions for SH modes decouple from rest of the motion, if the propagation occurs along an in-plane axis of symmetry. The stress-free boundary and rigid boundary are the dual situations that are cater by the equation obtained in the analysis by invoking appropriate boundary conditions on the plate surfaces. We also consider solutions that are bound near the surfaces in the limit d →∞. Some special cases have also been deduced and discussed. The results obtained theoretically have been verified numerically and illustrated graphically.

❖ FORMULATION

We consider an infinite generally- anisotropic plate of finite thickness *d*. The coordinate axes x_1, x_2, and x_3 of the model are chosen as to be analogous with the principal axes *x*, *y*, and *z* of the material, with *z* being normal to the plate. The mid plane of the plate is chosen to coincide with the x_1-x_2 plane. The bottom and upper surfaces of the thermoelastic plate are $x_3 = \pm d/2$, respectively, with $x_3 = 0$ being the mid-plane of the plate. The displacement field vector $\mathbf{u} = (u_1, u_2, u_3)$ satisfies equation of motion

$$\sum_{j=1}^{3}\left(\frac{\partial \tau_{ij}(\mathbf{u})}{\partial x_j}\right) = \rho\frac{\partial^2 u_i}{\partial t^2} \qquad i = 1,2,3 \tag{1}$$

where
$$\tau_{ij} = C_{ijkl} e_{kl}, \tag{2}$$

The summation convention is implied; ρ is the density, *t* is the time, u_i is the displacement in the x_i direction, σ_{ij} and e_{ij} are the stress and strain tensor respectively; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl}, \tag{3}$$

Strain-displacement relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \tag{4}$$

The displacement, stress components at the surface of the plate are:

$$D(x_3) = \{u_1, u_2, u_3) \tag{5}$$

$$S(x_3) = \{\tau_{13}, \tau_{23}, \tau_{33}) \tag{6}$$

Te boundary conditions at surfaces $x_3 = \pm h$

$$\sum_{j=1}^{3} \tau_{ij}(\mathbf{u}) m_j = 0 \qquad (\, i = 1,\, 2, \text{ and } 3\,) \tag{7}$$

ρ is the volume density, $(m_1, m_2, m_3) = (0, 0, \pm 1)$ are the normal vectors at the lower and upper surfaces, respectively, and $\tau_{ij}(\mathbf{u})$ $(i, j = 1, 2 \text{ and } 3)$ is the stress tensor.
Substituting equations (4) and (2) into equation (1), which is expressed by displacements as follows:

$$C_{jkl} u_{k,jl} = \rho \ddot{u}_i \;. \tag{8}$$

The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation.

❖ ANALYSIS

Assume that solutions to equation (9) are expressed by

$$u_j = U_j \exp[\imath\xi(\mathbf{n.x} - ct)], \quad j = 1, 2, 3 \tag{10}$$

where ξ is the wave number, $i = \sqrt{-1}$; c is the phase velocity (= ω/ξ); ω is the circular frequency, U_j are the constants related to the amplitudes of displacement, $\mathbf{n} = (n_1, n_2, n_3)$ are the components of the unit vector giving the direction of propagation. Substituting equation (10) into equation (9), this leads to the three coupled equations

$$\Pi_{ik} U_k = 0 \;, \tag{11}$$

where $\Pi_{ik} = \Gamma_k - \rho c^2 \delta_{ik}$, δ_{ik} is the Kronecker delta, and Γ_{ik} are the Christofied stiffness as follows:

$$\Gamma_{ik} = \Gamma_{ki} = C_{ijkl} n_j n_l \;. \tag{12}$$

For a nontrivial solution, we have

$$\det[\Pi_{ik}] = 0. \tag{13}$$

For an infinite anisotropic body, when n_i are given , three phase velocities can be obtained by eq. (13) For the finite thickness plate, we can obtain n_3 from equation (13), when n_1 and n_2 are given. On the other hand , equation ((13) can be written as polynomial equation of n_3 . The degree of this polynomial is six. We can solve it numerically and obtain eight roots $n_3 = \alpha_l$ $(l = 1, 2 \ldots . 6)$ and write the displacement as follows:

$$(u_1, u_2, u_3) = \sum_{l=1}^{6}(q_{1(l)}, q_{2(l)}, q_{3(l)}) B_l \exp(\imath\xi\alpha_l x_3)\exp[\imath\xi(n_1 x_1 + n_2 x_2 - ct)], \quad j=1, 2, 3. \tag{14}$$

where

$$q_{1(l)} = 1, \quad q_{2(l)} = \frac{\Gamma_{12}\Gamma_{13} - \Gamma_{23}\Pi_{11}}{\Gamma_{12}\Gamma_{23} - \Gamma_{13}\Pi_{22}}, \; q_{3(l)} = \left(\frac{-\Pi_{11}}{\Gamma_{13}}\right) - \left(\frac{\Gamma_{13}}{\Gamma_{12}}\right) q_{2(l)} \tag{15}$$

$$q_{2(l+1)} = q_{2(l)}, \; q_{3(l+1)} = -q_{3(l)}, \qquad l = 1,3,5.$$

Eqs. (14) satisfies eq. (9) and contains six undefined constants. From eqs. (14) and (4) strain tensor can be expressed as follows:

$$e_{ik} = \sum_{l=1}^{6} s_{ik_{(l)}} B_l \exp(\iota\xi\alpha_l x_3)\exp[\iota\xi(n_1x_1 + n_2x_2 - ct)] \tag{16}$$

where

$$s_{ik(l)} = \iota\xi(n_k q_{i(l)} + n_i q_{k(l)})/2, \; (i,k = 1,2,3). \tag{17}$$

The stress tensor are

$$\tau_k = \sum_{l=1}^{6} D_{k_{(l)}} B_l \exp(\iota\xi\alpha_l x_3)\exp[\iota\xi(n_1x_1 + n_2x_2 - ct)], \tag{18}$$

where

$$D_{ik(l)} = C_{jkl} s_{pq(l)} \qquad (i,k,p,q = 1,2,3). \tag{19}$$

In eqs. (17) and (19), $n_3 = \alpha_l$ $(l = 1,2....6)$. With eqs. (14) and (18), we have

$$u_j = \bar{u}_{j(l)} \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)], \tag{20}$$

$$\tau_{i3} = \bar{\tau}_{i3(l)} \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)], \tag{21}$$

where

$$\bar{u}_{j(l)} = \sum_{l=1}^{6} q_{j_{(l)}} \exp(\iota\xi\alpha_l x_3) B_l, \tag{23}$$

$$\bar{\tau}_{ik} = \sum_{l=1}^{6} D_{ik_{(l)}} \exp(\iota\xi\alpha_l x_3) B_l, \tag{24}$$

Eqs. (23)-(24) can be expressed as

$$\begin{bmatrix} \bar{u}_1, \bar{u}_2, \bar{u}_3 \\ \bar{\tau}_{13}, \bar{\tau}_{23}, \bar{\tau}_{33} \end{bmatrix} = \sum_{l=1}^{6} \begin{bmatrix} 1, \; q_{1(l)}, \; q_{2(l)} \\ D_{13(l)}, D_{23(l)}, D_{33(l)} \end{bmatrix} B_l \exp(i\xi\alpha_l x_3) \tag{25a}$$

In the matrix form

$$\begin{bmatrix} \bar{\tau}_{13} \\ \bar{\tau}_{23} \\ \bar{\tau}_{33} \\ \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} D_{13(1)} & D_{13(2)} & D_{13(3)} & D_{13(4)} & D_{13(5)} & D_{13(6)} \\ D_{21(1)} & D_{21(2)} & D_{21(3)} & D_{21(4)} & D_{21(5)} & D_{21(6)} \\ D_{31(1)} & D_{31(2)} & D_{31(3)} & D_{31(4)} & D_{31(5)} & D_{31(6)} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ q_{1(1)} & q_{1(2)} & q_{1(3)} & q_{1(4)} & q_{1(5)} & q_{1(6)} \\ q_{2(1)} & q_{2(2)} & q_{2(3)} & q_{2(4)} & q_{2(5)} & q_{2(6)} \end{bmatrix} \begin{bmatrix} U_{11}E_1 \\ U_{12}E_2 \\ U_{13}E_3 \\ U_{14}E_4 \\ U_{15}E_5 \\ U_{16}E_6 \end{bmatrix} \tag{25b}$$

which can be put in the block matrices form

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{25c}$$

where S and D are given by equations (8) and (9), A_1 and A_2 are given by

$$A_1 = \{U_{11}E_1, U_{12}E_2, U_{13}E_3\}, A_2 = \{U_{14}E_4, U_{15}E_5, U_{16}E_6\}. \tag{26}$$

On the upper and lower surfaces of the plate, eq. (25c) can be expressed as follows:
On the lower surface,

$$\begin{bmatrix} S \\ D \end{bmatrix}_{x_3 = -d/2} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{27}$$

On the upper surface,

$$\begin{bmatrix} S \\ D \end{bmatrix}_{x_3 = d/2} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{28}$$

where $G_i = K_i(x_3 = -d/2); Q_i = K_i(x_3 = d/2)(i = 1,...4)$.

According to eqs. (7) and (27), $\mathbf{A_2}$ can be expressed by $\mathbf{A_1}$.

$$\mathbf{A_2} = -(G_2)^{-1} G_1 \mathbf{A_1}. \tag{29a}$$

Substituting eq. (29a) into eq. (28), $S(d/2)$ and $D(d/2)$ can be expressed by $\mathbf{A}_1$.

$$S(d/2) = \Lambda_s \mathbf{A}_1, \tag{29b}$$

$$D(d/2) = \Lambda_d \mathbf{A}_1, \tag{29c}$$

where

$$\Lambda_s = Q_1 - Q_2(G_2)^{-1} G_1, \tag{29d}$$

$$\Lambda_d = Q_3 - Q_4(G_2)^{-1} G_1. \tag{29e}$$

With the boundary condition (8), we have

$$\Lambda_s \mathbf{A}_1 = \mathbf{0} \quad . \tag{29f}$$

For a non-trivial solution, the determinant of Λ_s must vanish.

$$\det(\Lambda_s) = 0. \tag{30}$$

The determinant of Λ_s is a function of phase velocity c and wavenumber ξ. When the wavenumber is assumed, the phase velocity can be obtained from eq. (30). Consequently, the dispersion of phase velocity is given by eq. (30) when the plate is free of stress. For any c and ξ which satisfy equation (30), the vector $\mathbf{A}_1$ can be determined. Then $\mathbf{A}_2$ can be obtained by (29a), then the displacements and stresses, can be determined easily.

❖ MATERIAL SYMMETRY

We have been dealing with the most general relationships applying to linear elastic, generally anisotropic materials. Such materials are referred to as triclinic materials. Many real materials have inherent symmetries which can greatly simplify their behaviour. In this section, we will look at some of these materials.

(i) Monoclinic materials are materials having, one plane of mirror symmetry. For monoclinic materials having x_1-x_2 as a plane of mirror symmetry, and x_1-x_3 as the plane of incidence, the equations of motion for monoclinic plate can be written as:

$$c_{11}u_{1,11} + c_{55}u_{1,33} + (c_{13}+c_{55})u_{3,13} = \rho\ddot{u}_1 - c_{16}u_{2,11} - c_{45}u_{2,33}$$

$$c_{16}u_{1,11} + c_{45}u_{1,33} + (c_{36}+c_{45})u_{3,13} = \rho\ddot{u}_2 - c_{66}u_{2,11} - c_{44}u_{2,33}$$

$$(c_{13}+c_{55})u_{1,13} + c_{55}u_{3,11} + c_{33}u_{33} = \rho\ddot{u}_3 - (c_{36}+c_{45})u_{2,13} \tag{31}$$

The use of solutions (15) in the form

$$(u_1, u_2, u_3) = (U_1, U_2, U_3)\exp[i\xi(n_1x_1 + n_2x_2 + n_3x_3 - ct), \qquad j = 1, 2, 3 \tag{32}$$

where (n_1, n_2 , n_3) = (sinθ , 0 , α) , θ is the angle of incidence, α is still an unknown parameters, U_1, U_2 , and U_3 are respectively the amplitudes of the displacements u_1 , u_2 and u_3 . Although solutions (32) are explicitly independent of x_2, an implicit dependence is contained in the transformation and the transverse displacement component u_2 is non-vanishing in eq. (32). The choice of solutions leads to three coupled eqs.

$$M_{mn}(\alpha)U_n = \mathbf{0} \qquad m,\ n = 1,2,3. \tag{33}$$

where $M_{11} = F_{11} + c_2\alpha^2$, $M_{12} = F_{12} + c_5\alpha^2$, $M_{13} = F_{13}\alpha$, $F_{11} = \sin^2\theta - \zeta^2$, $F_{12} = c_4\sin^2\theta$,

$M_{22} = F_{22} + c_6\alpha^2$, $M_{24} = F_{24}$, $F_{22} = c_3\sin^2\theta - \zeta^2$, $F_{23} = c_8\sin\theta\alpha$,

$M_{33} = F_{33} + c_1\alpha^2$, $M_{34} = F_{34}\alpha$, $F_{33} = c_2\sin^2\theta - \zeta^2$, $F_{13} = c_7\sin\theta$,

and

$$c_1 = c_{33}/c_{11},\ c_2 = c_{55}/c_{11},\ c_3 = c_{66}/c_{11},\ c_4 = c_{16}/c_{11},\ c_5 = c_{45}/c_{11},$$

$$c_6 = c_{44}/c_{11},\ c_7 = (c_{13}+c_{55})/c_{11},\ c_8 = (c_{36}+c_{45})/c_{11},\ \zeta^2 = c^2\rho/c_{11}. \tag{34}$$

The system of eqs. (33) has a non-trivial solution if the determinant of the coefficients of U_1, U_2, and U_3 vanishes, which yields a sixth-degree polynomial equation relating α to c, which can be written as

$$\alpha^6 + A_1\alpha^4 + A_2\alpha^2 + A_3 = 0. \tag{35}$$

where coefficients A_1 , A_2 and A_3 are

$$A_1 = (c_1c_6F_{11} - c_6F_{13}^2 + c_5F_{13}F_{23} - c_5^2F_{33} - c_2F_{23}^2 - 2c_1c_5F_{12} + c_5F_{21}F_{23} + c_2c_6F_{33})$$

$$A_2 = (F_{11}F_{23}^2 - F_{13}^2F_{22} + c_6F_{11}F_{33} + c_1F_{11}F_{22} - c_1F_{12}^2 + 2F_{12}F_{13}F_{23} + 2c_5F_{12}F_{33} + c_2F_{22}F_{33})$$

$$A_3 = (F_{11}F_{22} - F_{12}^2)F_{33} \qquad \Delta_E = (c_2c_6 - c_5^2)c_1.$$

This equation admits eight solutions for α (having the properties)

$$\alpha_2 = -\alpha_1,\quad \alpha_4 = -\alpha_3,\quad \alpha_6 = -\alpha_5 \quad . \tag{36}$$

Incorporating eqs. (36) in (15) to (28), and inspecting the resulting relations, we conclude that monoclinic symmetry implies further restrictions

$$q_{2(l+1)} = q_{2(l)},\quad q_{3(l+1)} = -q_{3(l)},$$

$$D_{33(l+1)} = D_{33(l)},\ D_{13(l+1)} = -D_{13(l)},\ D_{23(l+1)} = -D_{23(l)},\quad l = 1, 3, 5 \tag{37}$$

where now

$$D_{33(l)} = i\xi[(c_7 - c_2)\sin\theta + (c_8 - c_5)q_{2(l)} + c_1\alpha_l q_{3(l)l}],$$
$$D_{13(l)} = i\xi[c_2(\alpha_l + q_{3(l)}\sin\theta) + c_5\alpha_l q_{2(l)}],$$
$$D_{23(l)} = i\xi[c_5(\alpha_l + q_{3(l)}\sin\theta) + c_6\alpha_l q_{2(l)}]. \tag{38}$$

Substituting from eqs. (37) and (38) into (30), and then after algebraic manipulations and reductions to the monoclinic determinantal equation, it reduce and partitioned to a 2×2 diagonal matrix whose entries comprise of 3×3 square matrices. The determinant (30) can therefore be separated, leading to the two uncoupled characteristic equations

$$\sum_{k=1}^{3}(-1)^{k+1}D_{33(k)}G_k\tan^{\mp}(\gamma\alpha_k) = 0. \tag{39a,b}$$

corresponding to symmetric and anti symmetric modes of vibrations, respectively, with

$$G_1 = D_{13(3)}D_{23(5)} - D_{13(5)}D_{23(3)},\ G_3 = D_{13(1)}D_{23(5)} - D_{13(5)}D_{23(1)},\ G_5 = D_{13(1)}D_{23(3)} - D_{13(3)}D_{23(1)},\ \gamma = \xi d/2 = \omega d/2. \tag{40}$$

(ii) If we introduce a second plane of symmetry, we get an orthotropic material. In the case of orthotropic symmetry, we substitute from eqs. (41), which particularize the constitutive relations to orthotropic media, into the coefficients of the Appendix A. Inspection of the resulting entries leads to the conclusion that, for propagation along rotational symmetry axes, the matrix elements c_{16}, c_{26}, c_{36} and c_{45} also vanish. Of greatest importance is the fact that M_{12} and M_{23} in eq. (34) vanish. This means that SH wave motion uncouple from the rest of the motion. As a consequence, eq. (33), reduces to

$$c_1c_2\alpha^4 + (c_2F_{33} - c_1F_{11} - F_{13}^2)\alpha^2 + F_{11}F_{33} = 0 \tag{41}$$

and $$\alpha_7 = -\alpha_8 = \sqrt{(\zeta^2 - c_3\sin^2\theta)/c_6} \tag{42}$$

Notice that roots of (42) correspond to the SH motion, gives a purely transverse wave, this wave propagates without dispersion or damping. Equation (43) corresponds to the sagittal plane waves. As for the saggital plane motion we notice that for each α_l (l = 1,2 ...4) the displacements, stress amplitudes reduces to

$$q'_{3(l)} = -(F_{11} + c_2\alpha_l^2)/F_{13}\alpha_l, \tag{43}$$

$$D'_{33(l)} = [i\xi\{(c_7 - c_2)\sin\theta + c_1\alpha_l q'_{3(l)}\}] \tag{44}$$

$$D'_{13(l)} = i\xi[c_2(\alpha_l + q'_{3(l)}\sin\theta)]. \tag{45}$$

For the SH type wave, one now has

$$D_{23(6)} = -D_{23(5)} = c_6\alpha_5. \tag{46}$$

By employing the new relations (39), and following the matrix reduction steps used in obtaining the results of eqs. (39), one gets the reduced coupled characteristic equations

$$D'_{33(1)}D'_{13(3)}\tan^{\mp}(\gamma\alpha_1) - D'_{33(3)}D'_{13(1)}\tan^{\mp}(\gamma\alpha_3) = 0, \tag{47a, b}$$

$$\sin(2\gamma\alpha_5) = 0, \tag{48}$$

where γ is defined in eq. (40), eqs. (47a) and (47b), constitute the characteristic equations for symmetric and antisymmetric modes of vibrations, propagating along an in-plane axis of symmetry of a plate. Equation (48), corresponds to SH motion and studied in detail by [21]. The wave types uncouple since the wave vector is along the axis of symmetry. Furthermore, the relation (47a, b) implicitly contains corresponding results for material possessing higher than orthotropic symmetry. These include transversely isotropic, cubic, and isotropic. Here one needs only to exploit the appropriate restrictions on the elastic properties as described in eqs.

(iii) In Transversely isotropic, cubic and isotropic cases, Results for possessing transverse isotropy, can be easily obtained by noting the additional conditions imposed by symmetry, namely $c_{33} = c_{22}$, $c_{13} = c_{12}$, $c_{55} = c_{66}$, $2c_{44} = c_{22} - c_{23}$,

and $c_{11} = c_{22} = c_{33}$, $c_{12} = c_{13} = c_{23}$, $c_{44} = c_{55} = c_{66}$, (for cubic symmetry) (49)

$c_{11} = c_{33} = \lambda + 2\mu$, $c_{44} = \mu$, $c_{13} = \lambda$. (for the isotropic case)

❖ SURFACE WAVES

IN AN ARBITRARY DIRECTION OF A MONOCLINIC MATERIAL

In order to have surface wave, the roots α_i^2, $l=1,2,3$ of (35) must be either negative (so that square roots are purely imaginary) or complex numbers: this ensure that the superposition of partial waves has the properties of "exponential decay." There are two cases :

(i) α_i^2 i = 1,2,3 all are negative; and (ii) α_1^2, is negative $\alpha_2^2 = \alpha_3^{2*}$, are complex conjugates.

For the case (i), as d $\rightarrow\infty$, $\{\tan(\gamma\alpha_i)\}^{\pm} \rightarrow \pm i$ so that we have eq. (50a) from equations (39) :
For the case (ii), d $\rightarrow\infty$, $\{\tan(\gamma\alpha_i)\}^{\pm} \rightarrow \pm i$ and if $\alpha_2^2 = a+ib, \alpha_3^2 = a-ib$, $b>0$,
then $\{\tan(\gamma\alpha_l)\}^{\pm} \rightarrow \pm i$ and $\{\tan(\gamma\alpha_{l+1})\}^{\pm} \rightarrow \mp i$, thus equations (39) become eq. (50b)

$$D_{33(1)}G_1 + D_{33(3)}G_3 \pm D_{33(5)}G_5 = 0 . \quad (50a, b)$$

Surface wave velocity can be obtained by solving these equations.

PRINCIPAL DIRECTION (SAY X_1 DIRECTION)

We have two cases:

(i) α_i^2 , i = 1, 2 all are negative; and (ii) $\alpha_1^2 = \alpha_2^{2*}$, are complex conjugates. Equations (47a, b) become

$$D'_{33(1)}D'_{13(3)} \pm D'_{33(3)}D'_{13(1)} = 0 , \quad (51)$$

This equation reduces to the well known Rayleigh wave equation for isotropic media.

❖ FULLY CONSTRAINED BOUNDARY

If the boundary is fully constrained, then the boundary conditions are

$$D(0) = \mathbf{0} , \quad (52)$$

$$D(d) = \mathbf{0} . \quad (53)$$

Following the above procedure from equation (29) onwards, we obtain equation corresponding to constrained boundary

For a non-trivial solution, the determinant of Δ_d must vanish.

$$\det(\Delta_d) = 0 . \quad (54)$$

The determinant of Δ_d is a function of phase velocity c and wavenumber ξ. When the wavenumber is assumed, the phase velocity can be obtained from eq. (54). Consequently, the dispersion of phase velocity is given by eq. (54).

For any c and ξ which satisfy eq. (51), the vector $\mathbf{A}_1$ can be determined. Then $\mathbf{A}_2$ can be obtained by (29a), and then the displacement, as well as stress, can be determined easily.

❖ NUMERICAL DISCUSSION AND CONCLUSIONS

Numerical illustrations of the analytical characteristic equations are presented in the form of dispersion curves. In this section, the following non-dimensional for the phase velocity and wavenumber has been used throughout $\zeta = \dfrac{c}{\sqrt{c_{11}/\rho}}$, $\gamma = \xi d$.

Dispersion and damping curves are plotted by taking γ (wave number) real and letting ζ be complex, then the phase velocity is defined as c = Re(ζ)/ γ and the imaginary part of ζ is measure the damping of the waves. Numerical calculations of phase velocity are carried out from expression based on the dispersion relation in equation (47a) and (47b), for a beryl plate for which physical data is

C_{11} = 26.94, C_{33} = 23.63, C_{13} = 6.61, C_{44} = 6.54, ρ = 2.7

where elastic constants are measured in 10^{11} dynes/cm^2.

Dispersion curves for a beryl plate, where phase velocity ζ curves plotted as a function of wavenumber $\gamma(=\xi d)$ at four different values of an angle incidence θ (θ = 30 deg., 45 deg., 60deg. and 90 deg.) respectively. It is observed that dispersion curves corresponding to (47b) for symmetric and and to (47b) for antisymmetric wave modes get merged and then approach each others as wavenumber increases, where the phase tend towards the Rayleigh surface wave speed.

In Fig. 1 to Fig. 4, dispersion curves for a beryl plate where phase velocity curves plotted as a function of wavenumber ζ at four different values of an angle incidence θ (θ = 30 deg., 45 deg., 60deg. and 90 deg.) respectively. It is observed that for first symmetric vibration mode, the phase velocity decreases monotonically, and for antisymmetric vibration mode the phase velocity increases monotonically with increasing value of ξ (which is a characteristic of flexural waves), and is influenced by angle of incidence θ. It has also been observed from the figures that the point where the two modes cross each other decreases (in values of ζ), as the angle of incidence θ increases. Further, all the curves approaching each others at high ζ where the phase velocity tends towards the Rayleigh surface wave speed.

Each figure display three wave speeds corresponding to quasi-longitudinal, quasi-transverse and quasi-transverse at zero wavenumber limits, It is obvious that the largest value corresponds to the quasi-longitudinal mode. Higher modes appear in both cases as ζ increases. One of these seems to be associated with rapid change in the slope of the mode. Lower symmetric and antisymmetric modes are found more influenced at low values of wavenumber. Dispersion curves in Fig. 1, Fig. 3, Fig. 5and Fig. 7 and Fig. 4 corresponds to symmetric wave modes while Fig. 2, Fig. 4, Fig.6 and Fig. 8 to antisymmetric wave modes. The phase velocity of the lowest antisymmetric mode is observed to increase from zero value at zero wave number limits, whereas in the case of lowest symmetric mode it decreases from a value than and tends towards Rayleigh velocity asymptotically with an increase in wave number. The phase velocities of higher modes of propagation, symmetric and antisymmetric, attain quite large values at vanishing wave number.

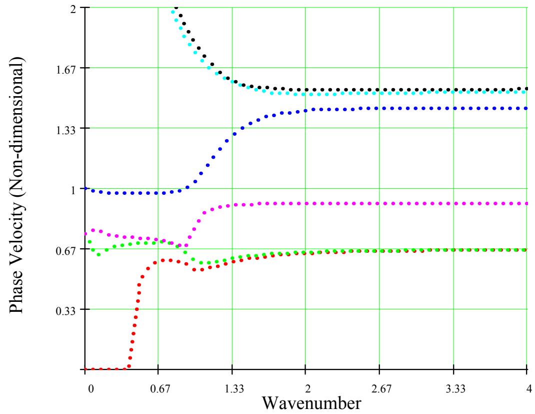

Figure 1. Dispersion curves of symmetric modes of beryl plate when, θ= 30 deg

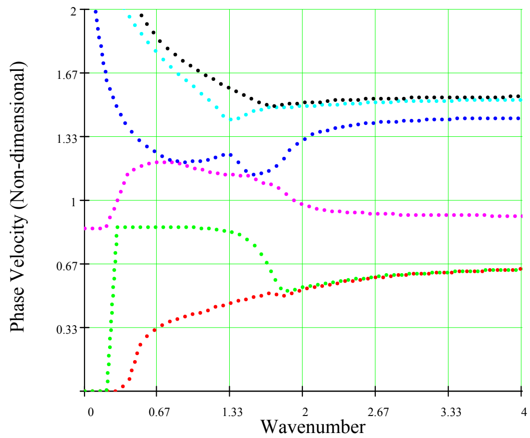

Figure 2. Dispersion curves of antisymmetric modes of beryl plate when, θ= 30 deg

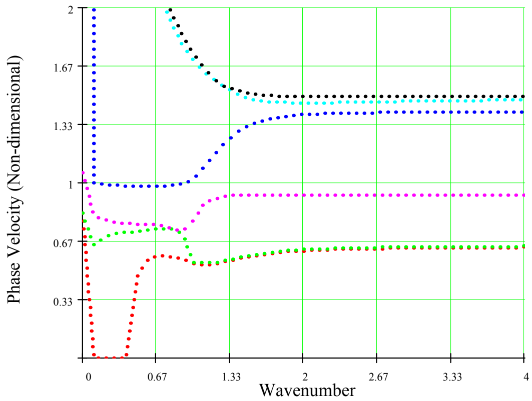

Figure 3. Dispersion curves of symmetric modes of beryl plate when, θ= 45 deg

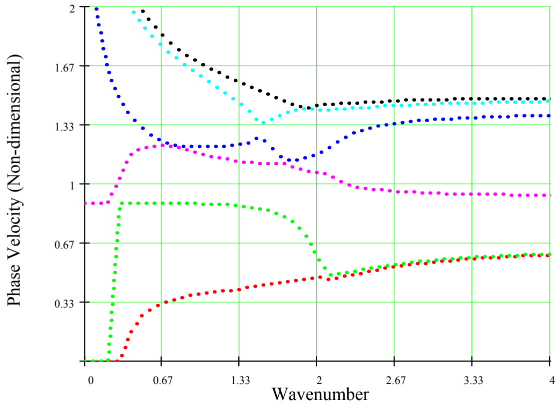

Figure 4. Dispersion curves of symmetric modes of beryl plate when, θ= 45 deg

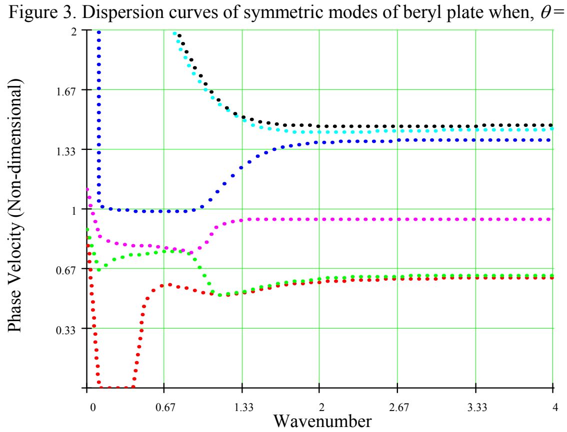

Figure 5. Dispersion curves of symmetric modes of beryl plate when, θ= 60 deg

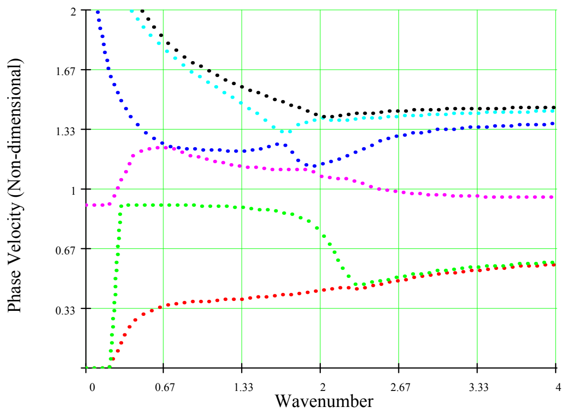

Figure 6. Dispersion curves of antisymmetric modes of beryl plate when, θ= 60 deg

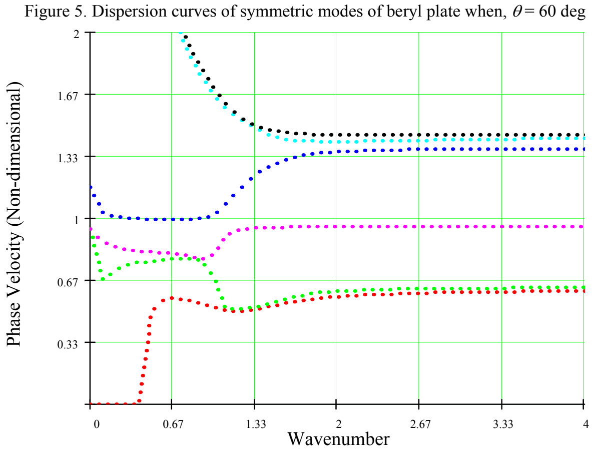

Figure 7. Dispersion curves of symmetric modes of beryl plate when, θ= 90 deg

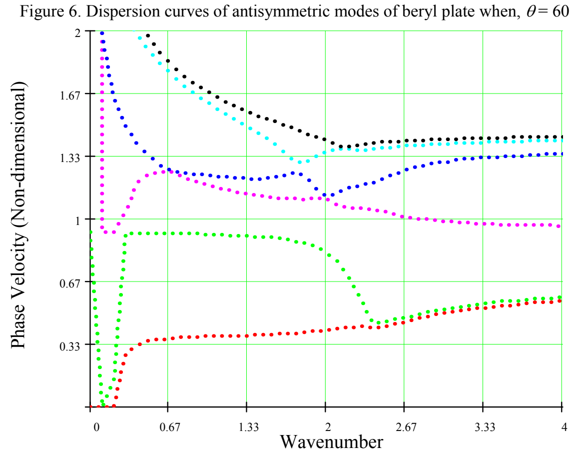

Figure 8. Dispersion curves of antisymmetric modes of beryl plate when, θ= 90 deg

Lowest antisymmetric and symmetric modes have zero velocity at vanishing wave numbers, but the phase velocity of these modes also become asymptotically close to the surface wave velocity with increasing value of the wave number. The behavior of higher modes of propagation is observed to be similar to other cases. The effect of incidence angle is observed to be small.

❖ CONCLUSIONS

The interaction of elastic waves has been investigated for anisotropic media. The horizontally polarized SH wave (48) gets decoupled from the rest of motion and propagates without dispersion or damping on the same plate. The other three waves namely, quasi-longitudinal (QL), quasi-transverse (QT) and Transverse (T-mode) of the medium are found coupled with each other due to anisotropic effects. The phase velocity of the waves gets modified due to the anisotropic effects. The dispersion characteristics for symmetric (extensional) and antisymmetric (flexural) have been taken into consideration.

❖ ACKNOWLEDGEMENTS

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