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A SIMPLE WAY TO ESTABLISH THE EQUATION OF SHELLS's YIELD SURFACE

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ABSTRACT: *A yield surface is a five-dimensional surface in the six-dimensional space of stresses. The yield surface is usually convex and the state of stress of inside the yield surface is elastic. When the stress state lies on the surface the material is said to have reached its yield point and the material is said to have become plastic. Further deformation of the material causes the stress state to remain on the yield surface, even though the surface itself may change shape and size. This is because stress states that lie outside the yield surface are non-permissible in rate-independent plasticity, though not in some models of viscoplasticity. This paper deals with the yield surface of shells.*
KEYWORDS: *yield surface, shells, elastic, plastic, viscoplasticity*

INTRODUCTION

Yield surface is usually expressed and visualized in three-dimensional space of principal stresses ($\sigma_1,\sigma_2,\sigma_3$), a two or three dimensional space extended by the constant stress (I_1,J_2,J_3) or a version of three-dimensional space that is Haigh-Westergaard stress. Yield surface is a frequent presence on Tresca yield surface, Huber-Von Mises yield surface, Mohr-Coulomb yield surface, Drucker-Prager yield surface, Bresler-Pister yield surface, Warnke-William yield surface, ... (Figure H1,H2&H3)

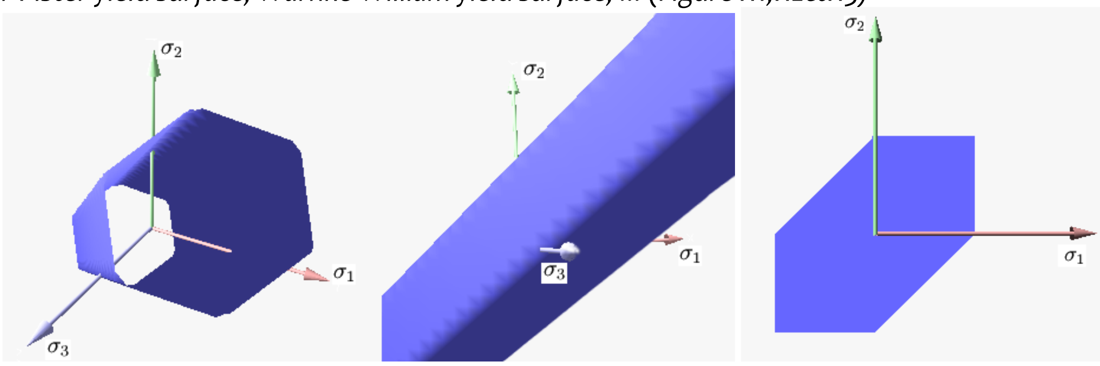

Figure H1. Tresca yield surface, 3D & 2D

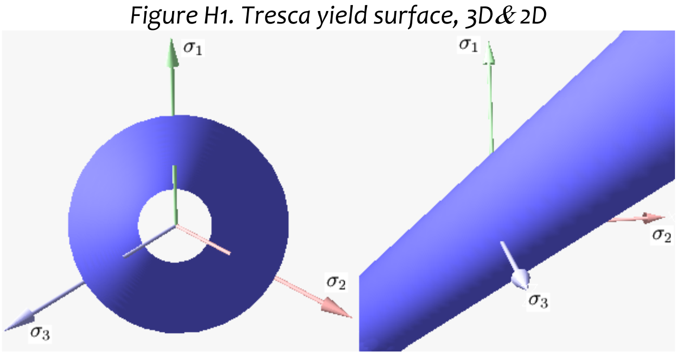

Figure H2. Huber-Von Mises yield surface

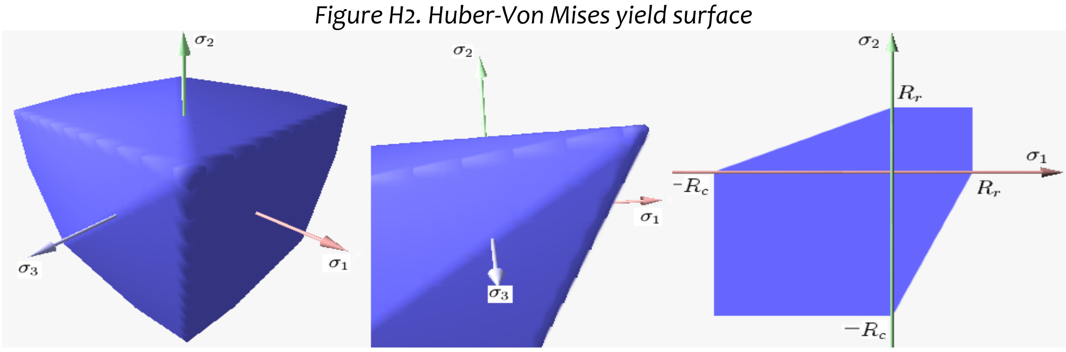

Figure H3. Mohr-Coulomb yield surface, 3D & 2D

RELATIONSHIP BETWEEN STRESS, DEFORMATION AND DISPLACEMENT

The internal forces in the shell that known as: N_θ, N_φ, M_θ, M_φ and Q.

Assumptions: (i) cross-section remains plane during deformation, (ii) the impact of negligible shear is ignored.

$$N_\theta = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\theta dz\,;\; N_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\varphi dz\,;\; M_\theta = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\theta z dz\,;\; M_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\varphi z dz$$

$$\varepsilon_\theta = \lambda_\theta + z\chi_\theta\,;\; \lambda_\theta = \frac{1}{r_2}(v\,\mathrm{cotg}\varphi - w);\; \chi_\theta = \frac{-\mathrm{cotg}\varphi}{r_1 r_2}(v + w')$$

$$\varepsilon_\varphi = \lambda_\varphi + z\chi_\varphi\,;\; \lambda_\varphi = \frac{1}{r_1}(v' - w);\; \chi_\varphi = -\frac{1}{r_1}\left[\frac{1}{r_1}(v + w')\right]$$

Equations:

$$\begin{cases} (r_0 N_\varphi)' - r_1 N_\theta \cos\varphi - r_0 Q + r_0 r_1 P_\varphi = 0 \\ (r_0 N_\varphi) + r_1 N_\theta \sin\varphi + (r_0 Q)' + r_0 r_1 P_r = 0 \\ (r_0 M_\varphi)' - r_1 M_\theta \cos\varphi - r_0 r_1 Q = 0 \end{cases}$$

YIELD CONDITION

Huber-Mises's condition:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2$$

Coulomb-Tresca's condition:

$$\sigma_{max} - \sigma_{min} = \sigma_o$$

One presented it in 3D and 2D (plane stress state, $\sigma_3 = 0$) as shown in Figure H4 and Figure H5

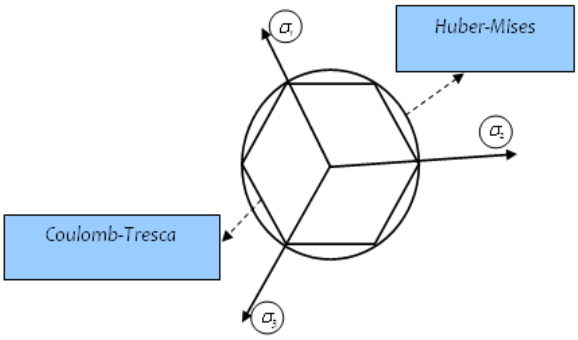

Figure H4. Stress state in 3D

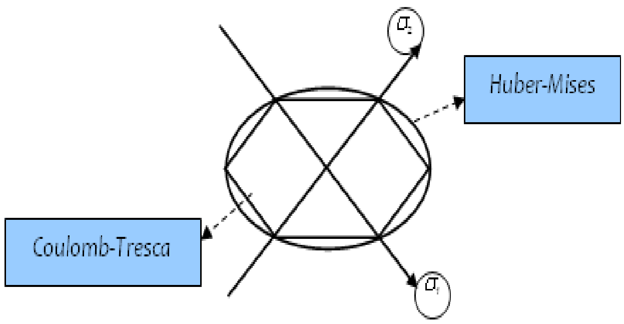

Figure H5. Plane stress state, σ_3=0, 2D

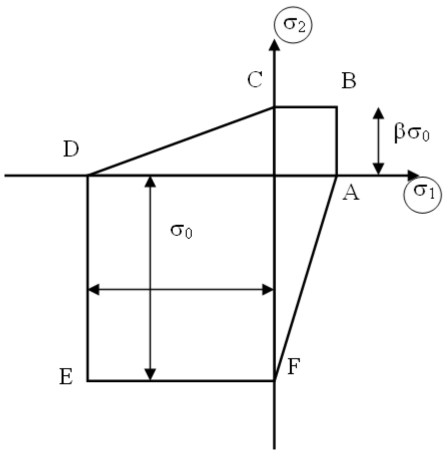

Figure H6. Isotropic material

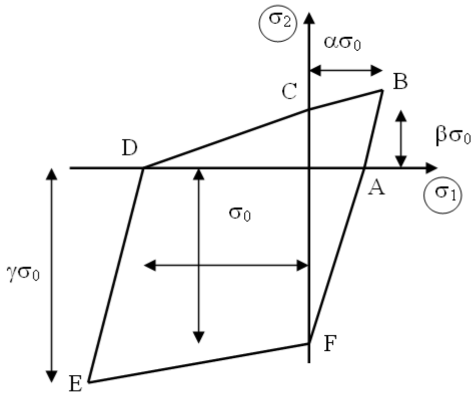

Figure H7. Orthotropic material

Easily find the Huber-Mises condition mentioned in the law nonlinear but Coulomb-Tresca condition mentioned simple linear law in the process of analysis for the shell. If one consider the shell material with yield stress difference between tension and compression, we will show in Figure H6 & H7. To simplify further, we can assume that the orthotropic property of material considered only in tension (σ_1, σ_2> 0). Then we reshow on Figure H8. For reference, we replace (φ, θ) by (1.2). Relationship between ε_i and z shown in Figure H10 & H11.

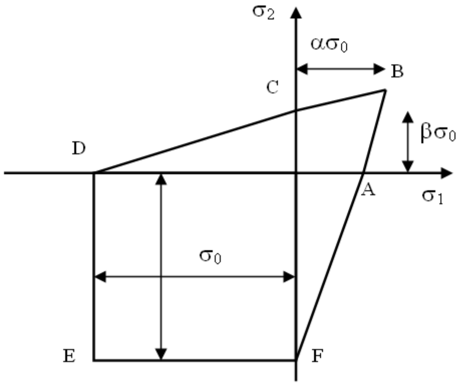

Figure H8. Simple model for orthotropic material

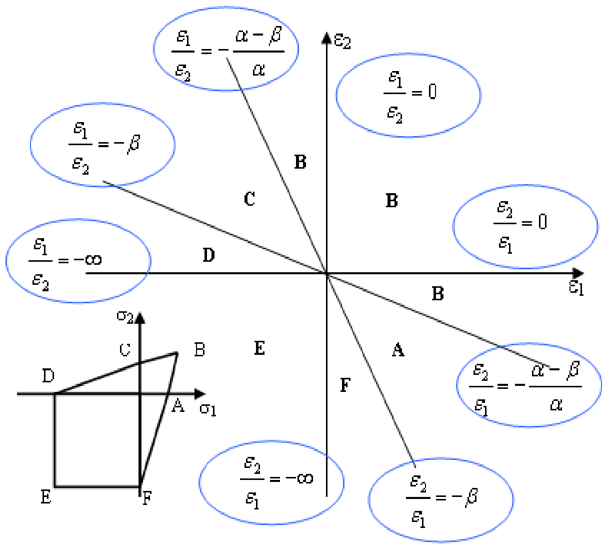

Figure H9. Simple analysis

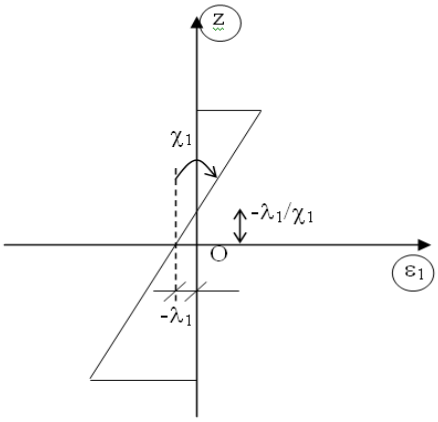

Figure H10. Relationship between ε_1 and z

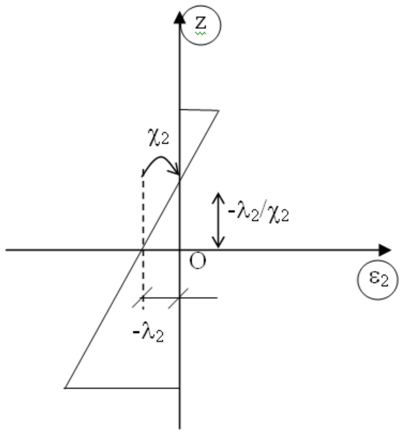

Figure H11. Relationship between ε_2 and z

ESTABLISHMENT EQUATION OF YIELD SURFACE

Relationship between ε_i, λ_i, χ_i and z:

$$\begin{cases} \varepsilon_1 = \lambda_1 + z\chi_1 \\ \varepsilon_2 = \lambda_2 + z\chi_2 \end{cases}$$

When the stress state is expressed as the edges of the yield surface, the legal condition is required perpendicular between (ε_1, ε_2) and these edges. At the corner, (ε_1, ε_2) will hold between two lines perpendicular to the two neighboring sides. One consider Figure H9, if the rate of deformation at a point depends on the first quadrant, one find the corresponding state of stress at this point is corner B on the yield surface. More specifically as follows

$$* \text{ if } \frac{-(\alpha-\beta)}{\alpha} \le \frac{\varepsilon_1}{\varepsilon_2} \le 0: \text{ stress state at corner B,}$$

$$* \text{ if } \frac{-(\alpha-\beta)}{\alpha} \ge \frac{\varepsilon_1}{\varepsilon_2} \ge -\beta: \text{ stress state at corner C,}$$

$$* \text{ if } -\beta \ge \frac{\varepsilon_1}{\varepsilon_2} \ge -\infty: \text{ stress state at corner D,}$$

One easily have

$$* \text{ if } \frac{h}{2} \ge z \ge -\frac{\lambda_2}{\chi_2};\ \begin{matrix} \varepsilon_1 \rangle 0 \\ \varepsilon_2 \rangle 0 \end{matrix}: \text{ regime B,}$$

$$* \text{ if } -\frac{\lambda_2}{\chi_2} \ge z \ge -\frac{\lambda_1}{\chi_1};\ \begin{matrix} \varepsilon_1 \rangle 0 \\ \varepsilon_2 \langle 0 \end{matrix} \qquad :$$

$$+\ 0 \ge \frac{\varepsilon_2}{\varepsilon_1} \ge -\frac{(\alpha-\beta)}{\alpha}: \text{ regime B,}$$

$$+\ -\frac{(\alpha-\beta)}{\alpha} \geq \frac{\varepsilon_2}{\varepsilon_1} \geq -\beta \ :\ \text{regime A,}$$

$$+\ -\beta \geq \frac{\varepsilon_2}{\varepsilon_1} \geq -\infty \ :\ \text{regime F,}$$

$$*\ \text{if}\ -\frac{\lambda_1}{\chi_1} \geq z \geq -\frac{h}{2}\ ;\ \begin{matrix}\varepsilon_1 \langle 0 \\ \varepsilon_2 \langle 0\end{matrix} :\ \text{regime E,}$$

Similarly one have,

$$*\ \text{if}\ \frac{h}{2} \geq z \geq -\frac{\lambda_2}{\chi_2}\ ;\ \begin{matrix}\varepsilon_1 \rangle 0 \\ \varepsilon_2 \rangle 0\end{matrix} \ :\ \text{regime B,}$$

$$*\ \text{if}\ -\frac{\lambda_2}{\chi_2} \geq z \geq -\frac{\lambda_1}{\chi_1};\ \begin{matrix}\varepsilon_1 \langle 0 \\ \varepsilon_2 \rangle 0\end{matrix}$$

$$+\ \text{if}\ 0 \geq \frac{\varepsilon_1}{\varepsilon_2} \geq -\frac{(\alpha-\beta)}{\alpha} \ :\ \text{regime B,}$$

$$+\ \text{if}\ -\frac{(\alpha-\beta)}{\alpha} \geq \frac{\varepsilon_1}{\varepsilon_2} \geq -\beta :\ \text{regime C,}$$

$$+\ \text{if}\ -\beta \geq \frac{\varepsilon_1}{\varepsilon_2} \geq -\infty \ :\ \text{regime D,}$$

$$*\ \text{if}\ -\frac{\lambda_1}{\chi_1} \geq z \geq -\frac{h}{2}\ ;\ \begin{matrix}\varepsilon_1 \langle 0 \\ \varepsilon_2 \langle 0\end{matrix} :\ \text{regime E,}$$

By using this equation: $\varepsilon_i = \lambda_i + z\chi_i$

$$\frac{h}{2} \geq z \geq -\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right] \ \text{respect to}\ \sigma_1 = \alpha\sigma_0$$

$$-\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right] \geq z \geq -\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right] \ \text{respect to}\ \sigma_1 = \beta\sigma_0$$

$$-\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right] \geq z \geq -\left[\frac{\lambda_1}{\chi_1}\right] \ \text{respect to}\ \sigma_1 = 0$$

$$-\left[\frac{\lambda_1}{\chi_1}\right] \geq z \geq -\left[\frac{h}{2}\right] \ \text{respect to}\ \sigma_1 = -\sigma_0$$

$$\frac{h}{2} \geq z \geq -\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right] \ \text{respect to}\ \sigma_2 = \alpha\sigma_0$$

$$-\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right] \geq z \geq -\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right] \ \text{respect to}\ \sigma_2 = 0$$

$$-\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right] \geq z \geq -\left[\frac{\lambda_1}{\chi_1}\right] \ \text{respect to}\ \sigma_2 = -\sigma_0$$

$$-\left[\frac{\lambda_1}{\chi_1}\right] \geq z \geq -\left[\frac{h}{2}\right] \ \text{respect to}\ \sigma_2 = -\sigma_0$$

In addition one have,

$$N_1 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_1 dz\ ;\ N_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_2 dz\ ;\ M_1 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_1 z dz\ ;\ M_2 = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_2 z dz$$

$$N_1 = \int_{-\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]}^{+\frac{h}{2}} \alpha\sigma_0 dz + \int_{-\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right]}^{-\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]} \beta\sigma_0 dz + \int_{-\left[\frac{\lambda_1}{\chi_1}\right]}^{-\left[\frac{(\beta)\lambda_1+\lambda_2}{(\beta)\chi_1+\chi_2}\right]} 0 dz + \int_{-\left[\frac{h}{2}\right]}^{-\left[\frac{\lambda_1}{\chi_1}\right]} -\sigma_0 dz$$

Similarly,

$$N_1=\sigma_0 h\left\{\frac{(\alpha-1)}{2}+\frac{\lambda_1}{h\chi_1}+\frac{\beta}{h}\left(\frac{\beta\lambda_1+\lambda_2}{\beta\chi_1+\chi_2}\right)+\frac{(\alpha-\beta)}{h}\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]\right\}$$

$$N_2=\sigma_0 h\left\{\frac{(\alpha-1)}{2}+\frac{1}{h}\left(\frac{\beta\lambda_1+\lambda_2}{\beta\chi_1+\chi_2}\right)+\frac{\alpha}{h}\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]\right\}$$

$$M_1=\frac{\sigma_0 h^2}{4}\left\{\frac{(\alpha+1)}{2}-2\left(\frac{\lambda_1}{h\chi_1}\right)^2-\frac{2\beta}{h^2}\left(\frac{\beta\lambda_1+\lambda_2}{\beta\chi_1+\chi_2}\right)^2-\frac{2(\alpha-\beta)}{h^2}\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]^2\right\}$$

$$M_2=\frac{\sigma_0 h^2}{4}\left\{\frac{(\alpha+1)}{2}-\frac{2}{h^2}\left(\frac{\beta\lambda_1+\lambda_2}{\beta\chi_1+\chi_2}\right)^2-\frac{2\alpha}{h^2}\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]^2\right\}$$

ILLUSTRATION

For case $-\frac{h}{2}\le-\frac{\lambda_1}{\chi_1}\le-\frac{(\lambda_1+\lambda_2)}{(\chi_1+\chi_2)}\le-\frac{\lambda_2}{\chi_2}\le\frac{h}{2}$ *(*)*

Presentation with dimensionless:

$$n_1=\frac{N_1}{\sigma_0 h};\ n_2=\frac{N_2}{\sigma_0 h};\ m_1=\frac{4M_1}{\sigma_0 h^2};\ m_2=\frac{4M_2}{\sigma_0 h^2}$$

$$f=-\frac{1}{h}\left(\frac{\beta\lambda_1+\lambda_2}{\beta\chi_1+\chi_2}\right);\ g=-\frac{1}{h}\left[\frac{(\alpha-\beta)\lambda_1+\alpha\lambda_2}{(\alpha-\beta)\chi_1+\alpha\chi_2}\right]$$

$$p=-\frac{\lambda_1}{h\chi_1};\ q=-\frac{1}{h}\left(\frac{\lambda_1+\lambda_2}{\chi_1+\chi_2}\right);\ r=-\frac{\lambda_2}{h\chi_2}$$

$$t=-\frac{1}{h}\left(\frac{\lambda_1+\beta\lambda_2}{\chi_1+\beta\chi_2}\right);\ k=-\frac{1}{h}\left[\frac{\alpha\lambda_1+(\alpha-\beta)\lambda_2}{\alpha\chi_1+(\alpha-\beta)\chi_2}\right]$$

Results are shown in Table 1 (*). The other cases are similarly done.

Table 1 (*).Results

Case	n_1	n_2	m_1	m_2
	$\frac{(\alpha-1)}{2}-p-(\alpha-\beta)g-\beta f$	$\frac{(\alpha-1)}{2}-f-\alpha g$	$\frac{(\alpha+1)}{2}-2\left[p^2+(\alpha-\beta)g^2+\beta f^2\right]$	$\frac{(\alpha+1)}{2}-2\left[f^2+\alpha g^2\right]$
	$-\frac{(\alpha-1)}{2}+f-\alpha g$	$\frac{(\alpha-1)}{2}+r+(\alpha-\beta)k+\beta t$	$-\frac{(\alpha+1)}{2}+2\left[t^2+\alpha k^2\right]$	$-\frac{(\alpha+1)}{2}+2\left[r^2+(\alpha-\beta)k^2+\beta t^2\right]$
	$\frac{(\alpha-1)}{2}-t-\alpha k$	$-\frac{(\alpha-1)}{2}-r-(\alpha-\beta)k-\beta t$	$\frac{(\alpha+1)}{2}-2\left[t^2+\alpha k^2\right]$	$\frac{(\alpha+1)}{2}-2\left[r^2+(\alpha-\beta)k^2+\beta t^2\right]$
	$\frac{(\alpha-1)}{2}+p+(\alpha-\beta)g+\beta f$	$\frac{(\alpha-1)}{2}+f+\alpha g$	$-\frac{(\alpha+1)}{2}+2\left[p^2+(\alpha-\beta)g^2+\beta f^2\right]$	$-\frac{(\alpha+1)}{2}+2\left[f^2+\alpha g^2\right]$
	$-p+t$	$\frac{(\beta-1)}{2}+r+\beta t$	$-2(p^2-t^2)$	$-\frac{(\beta+1)}{2}+2\left[r^2+\beta t^2\right]$
$-\frac{1}{2}\le p\le q\le r\le\frac{1}{2}$ (equation (*))	$\alpha k-(\alpha-\beta)g-\beta f$	$\frac{(\beta-1)}{2}-f+(\alpha-\beta)k-\alpha g$	$2\left[\alpha k^2-(\alpha-\beta)g^2-\beta f^2\right]$	$\frac{(\beta+1)}{2}+2\left[-f^2+(\alpha-\beta)k^2-\alpha g^2\right]$
	$-\alpha k+(\alpha-\beta)g+\beta f$	$\frac{(\beta-1)}{2}+f-(\alpha-\beta)k+\alpha g$	$\frac{(\alpha-1)}{2}-r-(\alpha-\beta)k-\beta t$	$-\frac{(\beta+1)}{2}+2\left[f^2-(\alpha-\beta)k^2+\alpha g^2\right]$
	$p-t$	$\frac{(\beta-1)}{2}-r-\beta t$	$2(p^2-t^2)$	$\frac{(\beta+1)}{2}-2\left[r^2+\beta t^2\right]$
	$\frac{(\beta-1)}{2}+t-(\alpha-\beta)g+\alpha k$	$\beta t+(\alpha-\beta)k-\alpha g$	$-\frac{(\beta+1)}{2}+2\left[t^2-(\alpha-\beta)g^2+\alpha k^2\right]$	$2\left[-\alpha g^2+(\alpha-\beta)k^2+\beta t^2\right]$
	$\frac{(\beta-1)}{2}-p-\beta f$	$-f+r$	$\frac{(\beta+1)}{2}-2\left[p^2+\beta f^2\right]$	$2(r^2-f^2)$
	$\frac{(\beta-1)}{2}+p+\beta f$	$t-r$	$-\frac{(\beta+1)}{2}+2\left[p^2+\beta t^2\right]$	$2(-r^2+t^2)$
	$\frac{(\beta-1)}{2}-t+(\alpha-\beta)g-\alpha k$	$-\beta t-(\alpha-\beta)k+\alpha g$	$\frac{(\beta+1)}{2}-2\left[t^2-(\alpha-\beta)g^2+\alpha k^2\right]$	$2\left[\alpha g^2-(\alpha-\beta)k^2-\beta t^2\right]$

CONCLUSIONS

When researching plastics for the shell we need to consider conditions for yield and yield surface of the shell in the case of orthotropic materials. This paper shows the simple way to have equation of yield surface of the shell for limit analysis method.

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