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[1.] K. L. VERMA

ANALYSIS OF THERMOELASTIC WAVES ON THE BASIS OF AN EXACT THEORY

[1.] DEPARTMENT OF MATHEMATICS GOVERNMENT POST GRADUATE COLLEGE HAMIRPUR (H.P.) 177005, INDIA

ABSTRACT: *In this article wave propagation in thin, flat, homogeneous thermoelastic plate of finite width and infinite length is investigated on the basis of an exact theory in the context of generalized theories of thermoelasticity. Frequency equations applicable to LS, GN models and classical theory are derived to investigate dispersion behavior of thermoelastic waves by invoking appropriate boundary conditions. Relevant results of previous investigations are deduced as special cases. The effects of the thermo-mechanical coupling, thermal relaxation times of the plate on the dispersion behavior are examined. Finally numerical solution of the frequency equation is carried out to present free wave dispersion curves for an aluminum plate.*
KEYWORDS: *Generalized thermoelasticity; thermomechanical coupling; without energy dissipation; Thermal relaxation time; attenuation*
***AMS Subject Classification**: 74A15, 74F05, 74H45, 74K20, 74L05*

INTRODUCTION

Study of thermally induced disturbances in thermoelastic plates is of interest for ultrasonic nondestructive evaluation of defects, materials characterization, and for dynamic response studies. Hence, the study of impact and wave propagation in thin plates that are weight-sensitive and depend upon temperature field, become very important.

The theory to include the effect of temperature change, known as the theory of thermoelasticity has been well established. Due to the coupling of thermal and strain fields, the theory is known as coupled theory of thermoelasticity. The basic governing equations of thermoelasticity in the usual framework of linear coupled thermoelasticity consist of the wave type (hyperbolic) equations of motion and the diffusion type (parabolic) equation of heat conduction. It is observed that a part of the solution of the energy equation tends to infinity. This implies that if an isotropic homogeneous elastic medium is subjected to thermal or mechanical disturbances, the effects in the temperature and displacement fields are felt at an infinite distance from the source of disturbance instantaneously. This implies that a part of the solution has an infinite velocity of propagation, which is physically impossible. To remedy this physically unrealistic contradiction, new theories based on a modified Fourier law of heat conduction or the incorporation of either an entropy production inequality or temperature rate-dependent constitutive variables were proposed Chandrasekharaiah[1], Ignaczak [2], Jakubowska [3]. Some researchers such as Kaliski [4], Lord and Shulman [5], Fox [6], Gurtin and Pipkin [7], Meixner[8] and Hetnarski and Ignaczak [9] have introduced the time needed for acceleration of the heat flow in the heat conduction equation along with the coupling between the temperature and strain fields. This new theory which is named as the 'Generalized Theory of thermoelasticity' eliminates the paradox of an infinite velocity of propagation and is based upon the more general linear functional relationship between the heat flow and the temperature gradients.

Of all the non-classical theories, the Lord and Shulman [5] model is in popular use engineering applications. The LS model introduces a single time constant to dictate the relaxation of thermal propagation, as well as the rate of change strain rate and the rate of change of heat generation. Ackerman et al.[10], Nayfeh and Nasser [11] have investigated the Maxwell's surface waves propagating along a half-space consisting of linearly elastic materials that conduct heat.

The theory of thermoelasticity without energy dissipation, recently proposed by Green and Naghdi [12], is one such theory (Here in after called GN theory). The discussion presented in Green and Naghdi includes the derivation of a complete set of governing equations of the linearized version of the theory for homogeneous and isotropic materials in terms of displacement and temperature fields and a proof of the uniqueness of the solution of the corresponding initial mixed boundary value problem. The uniqueness of the solution for an initial boundary value problem in this theory, formulated in terms of stress and energy-flux, has been established in Chandrasekharaiah [13].

Verma and Hasebe [14] discussed the propagation of thermoelastic vibrations in plates in the context of generalized theories of thermoelasticity. Mondal [15] obtained the frequency equations, corresponding to a thermoelastic plane wave in an infinite thermoelastic plate immersed in an infinite *liquid which is kept at uniform temperature, using thermoelastic potential, for symmetric and anti-symmetric vibrations about the vertical axis, taking into account the thermal relaxations.*

In this paper wave propagation in arbitrary thin, flat, homogeneous thermoelastic plate of finite width and infinite length is investigated on the basis of an exact theory, in the context of generalized theories [5, 12] of thermoelasticity. Frequency equations applicable to LS, GN models and classical theory are derived to investigate dispersion behavior of thermoelastic waves by invoking appropriate boundary conditions. Relevant results of previous investigations are deduced as special cases. The effects of the thermo-mechanical coupling, the relaxation times of the plate on the dispersion behavior are examined. Finally numerical solution of the frequency equation is carried out to present free wave dispersion curves for an aluminum plate.

BASIC GOVERNING EQUATIONS AND FORMULATION

Consider an infinite thermoelastic thin plate having thickness d, initially at uniform temperature T_0 such that its normal is aligned with x_3-axis of a reference Cartesian co-ordinate system $x_i=(x_1,x_2,x_3)$. The bottom surface of the plate is chosen to coincide with x_1-x_2 plane. The relations governing the plate are given as follows.

(a) The strain-displacement relations

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{1a}$$

(b) The stress-strain temperature relations

$$\sigma_{ij}=\lambda\delta_{ij}e_{kk}+2\mu e_{ij}-\gamma T\delta_{ij}, \tag{1b}$$

where δ_{ij} is the Kronecker delta.

(c) The governing field equations of motion and heat conduction in the context of generalized theory of thermoelasticity [5] are given by

$$\mu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_1 x_3}\right)=\rho\frac{\partial^2 u_1}{\partial t^2}+\gamma T_{,x_1} \tag{2a}$$

$$\mu\left(\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_3^2}\right)+(\lambda+\mu)\left(\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\frac{\partial^2 u_3}{\partial x_3^2}\right)=\rho\frac{\partial^2 u_3}{\partial t^2}+\gamma T_{,x_3} \tag{2b}$$

$$K\left(\frac{\partial^2 T}{\partial x_1^2}+\frac{\partial^2 T}{\partial x_3^2}\right)-\rho C_e\left(\frac{\partial T}{\partial t}+\tau_0\frac{\partial^2 T}{\partial t^2}\right)$$
$$=\gamma T_0\left[\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial t}\right)+\tau_0\left\{\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u_1}{\partial t^2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial^2 u_3}{\partial t^2}\right)\right\}\right] \tag{3}$$

where $\gamma=(3\lambda+2\mu)\alpha_t$, λ and μ are the Lame's parameters for isothermal deformations; α_t is the coefficient of thermal expansion; ρ, C_e and τ_0 are respectively the density, the specific heat at constant strain and the thermal relaxation time ; K is thermal conductivity of the medium.

ANALYSIS

If we now identify the plane of incidence to be the x_1x_3 plane, we propose a solution to eqs. (2) and (3) for the displacement u_j, $j=1,3$ and temperature T in the absence of body forces in the form

$$(u_1,u_3,T)=(U_1,U_3,U)e^{i\xi(x_1+\alpha x_3-ct)} \tag{4}$$

where ξ is the wave number , c is the phase velocity (= ω/ξ) ω is the circular frequency, α is the ratio of the x_3 and x_1 wave numbers. U_j and U (j = 1, 3) are the displacement amplitudes: and $i=\sqrt{-1}$. Substituting eq. (4) into eqs. (2) and (3), we obtain a characteristic equation relating α to c.

$$\alpha^6+B_1\alpha^4+B_2\alpha^2+B_3=0 \tag{5}$$

where

$$B_1=3-\zeta^2+\tau(1+\varepsilon_1)\zeta^2-\frac{\zeta^2}{c_2}$$

$$B_2=\left\{\left(3-2\zeta^2\right)+2\tau\left(1+\varepsilon_1\right)\zeta^2-\tau\zeta^4\right\}-\zeta^2\left\{(\tau-\zeta^2)+(1+\varepsilon_1)\tau\zeta^2\right\}\frac{1}{c_2}$$

$$B_3=\left\{(1-\zeta^2)+\tau(1+\varepsilon_1)\zeta^2-\tau\zeta^4\right\}\left\{1-\frac{\zeta^2}{c_2}\right\} \tag{6}$$

$\varepsilon_1\left(=\dfrac{\gamma^2 T_0}{\rho C_e(\lambda+2\mu)}\right)$ *is a thermoelastic coupling constant; and*

$$\tau=\tau_0+\frac{i}{\omega},\quad \zeta^2=\frac{\rho c^2}{(\lambda+2\mu)},\quad c_2=\frac{\mu}{(\lambda+2\mu)}. \tag{7}$$

This equation admits six solutions for α (having the properties $\alpha_2 = -\alpha_1$, $\alpha_4 = -\alpha_3$, $\alpha_6 = -\alpha_5$) and by using superposition results, the displacements and temperature are as follows:

$$u_j = \sum_{\ell=1}^{6} q_{j(\ell)} e^{\iota\xi\alpha_\ell x_3} e^{\iota\xi(x_1-ct)} A_\ell \tag{8}$$

$$T = \sum_{\ell=1}^{6} \Theta_\ell e^{\iota\xi\alpha_\ell x_3} e^{\iota\xi(x_1-ct)} A_\ell \quad j = 1,3 \tag{9}$$

where
$$q_{1(\ell)} = 1 \ , \ \ell = 1,2...6 \tag{10}$$

$$q_{3(\ell)} = \alpha_\ell \ , \ q_{3(3)} = {-1}/{\alpha_3} \quad \ell = 1,5$$

$$q_{3(m+1)} = -q_{3(m)} \ , \quad m = 1,3,5 \tag{11}$$

$$\Theta_\ell = \frac{\varepsilon_1\omega_1\tau\zeta^2}{\left(1+\alpha_\ell^2 - \tau\omega_1\zeta^2\right)}\left(1+\alpha_\ell q_{3(\ell)}\right), \ \omega_1 = \frac{C_e(\lambda+2\mu)}{K}, \Theta_3 = 0, \ \ell = 1,5$$

$$\Theta_{m+1} = \Theta_m \ . \qquad m = 1,3,5 \ . \tag{12}$$

The stresses and temperature gradient are

$$\sigma_{j3} = \sum_{\ell=1}^{6} r_{j(\ell)} e^{i\xi\alpha_\ell x_3} e^{i\xi(x_1-ct)} A_\ell \tag{13}$$

$$\frac{\partial T}{\partial x_3} = \sum_{\ell=1}^{6} \Omega_\ell e^{i\xi\alpha_\ell x_3} e^{i\xi(x_1-ct)} A_\ell \tag{14}$$

where
$$r_{1(\ell)} = 2\mu\alpha_\ell, \quad r_{1(3)} = \frac{\mu\left(\alpha_3^2-1\right)}{\alpha_3} \quad \ell = 1,5$$

$$r_{1(m+1)} = -r_{1(m)} \ , \ m = 1,3,5 \tag{15}$$

$$r_{3(1)} = [\mu\{\zeta^2/c_2 - 2\}]$$

$$r_{3(3)} = 2\mu, \quad r_{3(1)} = r_{3(5)} \ , r_{3(m+1)} = r_{3(m)}, \ m = 1,3,5 \ . \tag{16}$$

$$\Omega_\ell = \iota\xi\alpha_\ell \quad \ell = 1,5 \ , \ \Omega_3 = 0$$

$$\Omega_{m+1} = \Omega_m \quad m = 1,3,5 \tag{17}$$

With eqs. (8) and (9), we have

$$u_j = \overline{u}_{j(\ell)} e^{i\xi(x_1-ct)} \ , \tag{18}$$

$$T = \overline{T} e^{i\xi(x_1-ct)} \ . \tag{19}$$

With eqs. (13) and (14) , we have

$$\sigma_{j3} = \overline{\sigma}_{j3} e^{i\xi(x_1-ct)} \ , \ j = 1,3 \tag{20}$$

$$\frac{\partial T}{\partial x_3} = \frac{\partial \overline{T}}{\partial x_3} e^{i\xi[x_1-ct]} \tag{21}$$

where
$$\overline{u}_{j(\ell)} = \sum_{\ell=1}^{6} q_{j(\ell)} e^{\iota\xi\alpha_\ell x_3} A_\ell \tag{22}$$

$$\overline{T} = \sum_{\ell=1}^{6} \Theta_\ell e^{\iota\xi\alpha_\ell x_3} A_\ell \tag{23}$$

$$\overline{\sigma}_{j3} = \sum_{\ell=1}^{6} r_{j(\ell)} e^{\iota\xi\alpha_\ell x_3} A_\ell \ , \tag{24}$$

$$\frac{\partial \overline{T}}{\partial x_3} = \sum_{\ell=1}^{6} \Omega_\ell e^{i\xi\alpha_\ell x_3} A_\ell \ . \tag{25}$$

Eqs. (22)-(25) can be expressed by the following matrix form

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 & \mathbf{F}_2 \\ \mathbf{F}_3 & \mathbf{F}_4 \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \tag{26}$$

where

$$\mathbf{S}(x_3) = \left\{ \bar{\sigma}_{13}, \bar{\sigma}_{33}, \frac{\partial \bar{T}}{\partial x_3} \right\}, \tag{27}$$

$$\mathbf{D}(x_3) = \{\bar{u}_3, \bar{u}_3, \bar{T}\},, \tag{28}$$

$$\mathbf{A}_1 = \{A_1, A_2, A_3\}, \tag{29}$$

$$\mathbf{A}_2 = \{A_4, A_5, A_6\}, \tag{30}$$

and $\mathbf{F}_j$ are 3×3 matrices $j = 1, 2, 3, 4$.

On the upper and the bottom surfaces of the plate, eq. (26) can be expressed as follows:

- On the lower surface:

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{D} \end{pmatrix}_{x_3=0} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_3 \\ \mathbf{P}_2 & \mathbf{P}_4 \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \tag{31}$$

- On the upper surface:

$$\begin{pmatrix} \mathbf{S} \\ \mathbf{D} \end{pmatrix}_{x_3=d} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix} \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \tag{32}$$

where $\mathbf{P}_j = \mathbf{F}_j$ at $x_3 = 0$ and $\mathbf{Q}_j = \mathbf{F}_j$ at $x_3 = d$ $j = 1, 2, 3, 4$. (33)

Eqs. (31) and (32) can now be used to present solutions for a variety of situations. In the first we consider a free plate in the context of generalized thermoelasticity.

STRESS-FREE BOUNDARY

If the boundary conditions are that the stresses on the surfaces of the plate vanish i.e. plate is free of stress, then from (27), we have

$$\mathbf{S}(0) = \mathbf{0}, \tag{34}$$

$$\mathbf{S}(d) = \mathbf{0} \tag{35}$$

According to eqs. (34) and (31), $\mathbf{A}_2$ can be expressed by $\mathbf{A}_1$.

$$\mathbf{A}_2 = -(\mathbf{P}_2)^{-1}\mathbf{P}_1\mathbf{A}_1. \tag{36}$$

On substituting eq. (36) into eq. (32), $\mathbf{S}(d)$ and $\mathbf{D}(d)$ can be expressed by $\mathbf{A}_1$.

$$\mathbf{S}(d) = \Lambda_s\mathbf{A}_1, \tag{37}$$

$$\mathbf{D}(d) = \Lambda_d\mathbf{A}_1, \tag{38}$$

where

$$\mathbf{\Lambda}_s = \mathbf{Q}_1 - \mathbf{Q}_2(\mathbf{P}_2)^{-1}\mathbf{P}_1, \tag{39}$$

$$\mathbf{\Lambda}_d = \mathbf{Q}_3 - \mathbf{Q}_4(\mathbf{P}_2)^{-1}\mathbf{P}_1. \tag{40}$$

With the boundary condition (35), we have

$$\mathbf{\Lambda}_s\mathbf{A}_1 = 0. \tag{41}$$

For a non-trivial solution, the determinant of Λ_s must vanish.

$$\det(\mathbf{\Lambda}_s) = 0. \tag{42}$$

The determinant of Λ_s is a function of phase velocity c and wave number ξ. When the wave number is assumed, the phase velocity can be obtained from eq. (42). consequently, the dispersion of phase velocity is given by eq. (42) when the plate is free of stress.

For any c and ξ which satisfy eq. (42), the vector $\mathbf{A}_1$ can be determined. Then $\mathbf{A}_2$ can be obtained by (36), and then the displacement, temperature as well as stress, can be determined easily.

Classical case:

This case corresponds to the situations when the strain and temperature fields are not coupled with each other. In this case the thermo-mechanical coupling constant ε_1 identically zero. Then eq. (42), after straight forward calculations become product of two period equations for the symmetric and antisymmetric modes, respectively, for a free homogeneous isotropic plate of thickness `d` [16].

Another important situation is that of a constrained boundary of thermoelastic plate in the context of generalized thermoelasticity.

CONSTRAINED BOUNDARY

If the boundary is fully constrained, then the boundary conditions are

$$D(0) = 0, \tag{43}$$

$$D(d) = 0. \tag{44}$$

According to eqs. (43) and (31), $\mathbf{A}_2$ can be expressed by $\mathbf{A}_1$.

$$\mathbf{A}_2 = -(\mathbf{P}_4)^{-1}\mathbf{P}_3\mathbf{A}_1. \tag{45}$$

Substituting eq. (45) into eq. (32), $S(d)$ and $D(d)$ can be expressed by $\mathbf{A}_1$.

$$S(d) = \Delta_s \mathbf{A}_1, \tag{46}$$

$$D(d) = \Delta_d \mathbf{A}_1, \tag{47}$$

where

$$\mathbf{\Delta}_s = \mathbf{Q}_1 - \mathbf{Q}_2(\mathbf{P}_4)^{-1}\mathbf{P}_3 \ , \tag{48}$$

$$\mathbf{\Delta}_d = \mathbf{Q}_3 - \mathbf{Q}_4(\mathbf{P}_4)^{-1}\mathbf{P}_3 \ . \tag{49}$$

With the boundary condition (44), we have

$$\mathbf{\Delta}_d \mathbf{A}_1 = \mathbf{0} \ . \tag{50}$$

For a non-trivial solution, the determinant of Δ_d must vanish.

$$\det(\mathbf{\Delta}_d) = 0 . \tag{51}$$

The determinant of $\mathbf{\Delta}_d$ is a function of phase velocity c and wavenumber ξ. When the wavenumber is assumed, the phase velocity can be obtained from eq. (51). Consequently, the dispersion of phase velocity is given by eq. (51).

For any c and ξ which satisfy eq. (51), the vector $\mathbf{A}_1$ can be determined. Then $\mathbf{A}_2$ can be obtained by (45), and then the displacement, temperature as well as stress, can be determined easily.

A third important situation is that solutions which are bound near the surfaces in the limit $d\to\infty$ in the context of generalized thermoelasticity.

SURFACE WAVE DETERMINATION

In order to have a surface wave, roots α_j^2 (j = 1,2,3) of eq. (5), must be either negative (so that square roots are purely imaginary) or complex numbers; this ensures that the superposition of partial waves has the property of exponential decay. For these cases, as $d\to\infty$, then eq. (42) reduces to

$$\left(1-\alpha_3^2\right)^2\left(\alpha_1^2+\alpha_5^2+\alpha_1\alpha_5+1-\zeta^2\right)-4\alpha_1\alpha_3\alpha_5(\alpha_1+\alpha_5)=0 \tag{52}$$

where α_1^2, α_3^2 and α_5^2 are roots of (5). Equation (52) is the same as obtained and discussed by[11].

When $\varepsilon_1 = 0$ after lengthy and straight forward calculation, we have

$$\left(2-\frac{\zeta^2}{c_2}\right)^4 = 16\left(1-\frac{\zeta^2}{c_2}\right)(1-\zeta^2) . \tag{53}$$

This reveals that the elastic waves will be non-dispersive in character in this case, which is in agreement with Stonely [17] in the non-dimensional case.

COUPLED THERMOELASTICITY

This case corresponds to no thermal relaxation time, i.e. $\tau_0 = 0$ and hence $\tau = 1/\omega$. Proceeding on the same lines as in the above case, we again arrived at eq. (52). This is in agreement with the corresponding results obtained by, [18, 19, 20]. If we use the condition $\omega \ll 1$, then the eq. (52) reduces to

$$(1+\varepsilon_1)\left(2-\frac{\zeta^2}{c_2^2}\right)=16\left\{(1+\varepsilon_1)-\zeta^2\right\}\left(1-\frac{\zeta^2}{c_2^2}\right). \tag{54}$$

when $\varepsilon_1=0$, eq. (54) becomes eq. (53) and for $\varepsilon_1\neq 0$, it corresponds to the results obtained by [19, 20].

THERMOELASTICITY WITHOUT ENERGY DISSIPATION

The fundamental equations for such a medium, with heat sources and body forces absent, in the context of generalized thermoelasticity developed by Green and Naghdi [12], are given by

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla\,\mathrm{div}\boldsymbol{u} - \gamma\nabla\theta = \rho\ddot{\boldsymbol{u}}, \tag{55}$$

$$\rho C\,\ddot{\theta} + \gamma\theta_0\,\mathrm{div}\,\ddot{\boldsymbol{u}} = k^*\,\nabla^2\theta. \tag{56}$$

Here $\mathbf{u}(x,z,t)=(u,0,w)$ is the displacement vector; θ is the temperature change above the uniform reference temperature θ_0; ρ is the mass density; C is the specific heat at constant deformation; λ and μ are the LamÉ's parameters; $\gamma=(3\lambda+2\mu)\beta^*$ β^* is the coefficient of volume expansion; and k* is a material constant characteristic of the theory.

The strain tensor **E** and the stress tensor **T** associated with u and θ are given by the following geometrical and constitutive relations, respectively, as

$$E = \tfrac{1}{2}\,[\nabla u + \nabla u^T], \tag{57}$$

$$\vec{\sigma} = \lambda(div\ \ u)\, I \ + \ \mu(\nabla u + \nabla u^T) - \gamma\theta I\,. \tag{58}$$

In all the above equations, the direct vector/ tensor notation [14] is employed; also, an overdot denotes the partial derivative with respect to the time variable t. Some of the symbols and the notations used here are slightly different from those employed in [14]. We suppose that the constants appearing in eqs. (55) and (56) satisfy the inequalities

$$\mu > 0, \lambda + 2\mu > 0, \rho > 0, \theta_0 > 0, C > 0, k^* > 0 \tag{59}$$

Equations (55) and (56) represent a fully hyperbolic system that permits finite speeds for both elastic and thermal disturbances, which are coupled together in general.

Define the dimensionless quantities

$$x' = \frac{1}{l}x,\ t' = \frac{v}{l}t,\ u' = \frac{1}{l}\,\frac{(\lambda+2\mu)}{\gamma\theta_o}\,u\,,$$

$$\theta' = \frac{\theta}{\theta_o}\,,\ E' = \frac{(\lambda+2\mu)}{\gamma\theta_o}E,\ T' = \frac{1}{\gamma\theta_o}T\,. \tag{60}$$

Here l is a standard length and v is the standard speed

Introducing eq. (60) into eqs. (55) and (56) and suppressing the primes, we obtain the following

$$\vec{E} = \tfrac{1}{2}\left[\nabla u + \nabla u^T\right] \tag{61}$$

$$\vec{\sigma} = \left(1 - 2\frac{C_2^2}{C_1^2}\right)(div\ \vec{u})\ I + \frac{C_2^2}{C_1^2}\left[\nabla u + \nabla u^T\right] - \theta I \tag{62}$$

Here $$C_1^2 = \frac{\lambda+2\mu}{\rho v^2} \quad C_2^2 = \frac{\mu}{\rho v^2} \quad C_3^2 = \frac{K^*}{Cv^2} \quad \varepsilon_1 = \frac{v^2\theta_o}{C(\lambda+2u)} \tag{63}$$

$$C_2^2\ \nabla^2\ \vec{u} + (C_1^2 - C_2^2)\ \nabla\ div\,\vec{u} - C_1^2\ \nabla\theta = \ddot{\vec{u}} \tag{64}$$

$$C_3^2\ \nabla^2\theta = \ddot{\theta} + \varepsilon_1\nabla\,\ddot{\vec{u}} \tag{65}$$

The eqs. (64) and (85) serve as a coupled system of governing equations for the non-dimensional field $\vec{u}$ and non-dimensional temperature θ. We observe that C_1 and C_2 respectively represent the non -dimensional speeds of purely elastic dilatational and shear waves and that C_3 represents the non - dimensional speed of purely thermal waves. Also ε_1 is the usual thermoelastic coupling parameter [14]. With this choice of co-ordinate system, eqs. (55) and (56) in the Green and Naghdi [12] theory relevant to our problem becomes

$$C_1^2 u_{1,11} + C_2^2 u_{1,33} + (C_1^2 - C_2^2)u_{3,13} - C_1^2 T_{,1} = \ddot{u}_1, \tag{66}$$

$$C_1^2 u_{3,33} + C_2^2 u_{3,33} + (C_1^2 - C_2^2)u_{1,13} - C_1^2 T_{,3} = \ddot{u}_3, \tag{67}$$

$$C_1^2(T_{,11} + T_{,33}) = \ddot{T} + \varepsilon_1(\ddot{u}_{1,1} + (\ddot{u}_{3,3})\,. \tag{68}$$

Substituting u_1, u_3, and T from (4) into eqs. (66) to (68) , we obtain a system of coupled equations

$$M_{pq}(\alpha_q) = 0, \qquad p,q = 1,3,4 \tag{69}$$

with corresponding coefficients as

$$M_{11} = C_2\alpha^2 + C_1 - c^2,\ M_{13} = (C_1^2 - C_2^2)\alpha\,,\ M_{14} = C_1^2, M_{13} = M_{31},$$
$$M_{33} = C_2^2 + C_1^2\alpha^2 - c^2, M_{34} = C_1^2\alpha\,, M_{41} = c^2\varepsilon_1,\ M_{43} = \varepsilon_1 c^2\alpha\,,$$
$$M_{44} = C_3^2(1+\alpha^2) - c^2. \tag{70}$$

The existence of nontrivial solution for U_1, U_2 and U demands vanishing of the determinant in eqs. (69) for GN theory, and yields the polynomial equation

$$[C_3^2(\alpha^2+1) - c^2][\alpha^4 + P\alpha^2 + Q] = 0\ , \tag{71}$$

where

$$P = \frac{\{2C_1^2C_3^2 - [(1+\varepsilon_1)C_1^2 + C_3^2]c^2\}}{C_1^2C_3^2},\ Q = \frac{\{(c^4 - [(1+\varepsilon_1)C_1^2 + C_3^2]c^2 + C_1^2C_3^2\}}{C_1^2C_3^2}. \tag{72}$$

Here α_1^2, α_5^2 are roots of the equation $\alpha^4 + P\alpha^2 + Q = 0$ and are corresponds to coupled longitudinal and thermal waves in GN theory, whereas $\alpha_3^2 = \dfrac{c^2}{C_2^2} - 1$ corresponds to transverse wave which is not affected by the temperature fields, thermal relaxations and thermo-mechanical coupling.

Solving eq. (69) for the six roots of α and using superposition results in the following formal solution relating the displacements, temperature, thermal stresses and temperature gradient in the context of thermoelasticity without energy dissipation to its wave amplitudes, we get the following relations corresponding to eqs. (10) to (12) of Section 3 as:

$$r_{1(1)} = \frac{C_2^2}{C_1^2}\left(\frac{c^2}{C_2^2} - 2\right), \quad r_{3(3)} = -2\frac{C_2^2}{C_1^2}, \; r_{1(5)} = r_{1(1)}, \; r_{1(1)} = 2\frac{C_2^2}{C_1^2}\alpha_1, \; r_{1(3)} = \frac{C_2^2}{C_1^2}\frac{[\alpha_3^2 - 1]}{\alpha_3}, \quad r_{1(5)} = 2\frac{C_2^2}{C_1^2}\alpha_2,$$

$$\Omega_1 = \alpha_1\Theta_1, \quad \Theta_3 = 0, \; \Omega_5 = \alpha_2\Theta_5$$

$$\Theta_q = \frac{c^2\varepsilon_1[(1+\alpha_q^2)C_2^2 - c^2]}{[C_1^2\varepsilon_1 c^2 - \{C_3^2(1+\alpha_q^2) - c^2\}\{(C_1^2 - C_3^2)\}]}, \quad q = 1,5$$

$$\bar{\sigma}_{33} = \frac{\sigma_{33}}{i\xi}, \; \bar{\sigma}_{13} = \frac{\sigma_{13}}{i\xi}, \; \bar{T}' = \frac{T'}{i\xi}. \tag{73}$$

$$E_q = e^{i\xi\alpha_q z}, \; E = e^{i\xi(x-ct)}, \quad q = 1,2,\cdots 6$$

The various parameters α_1, α_2, α_3, Ω_1, Ω_2, $r_{1(j)}$ and $r_{3(j)}$, $j = 1,3,5$ etc. are then specialized in the context of thermoelasticity without energy dissipation to obtain the results corresponds to (26) and then to (31) and (32).

Further, introducing the appropriate stress free and constrained boundary conditions on the corresponding stresses and temperature gradient relevant to our problem, in this theory, and proceeding as in the previous section, we obtained the relations, which are of the same form as (42) and (51) (of Section 3) in the theory of thermoelasticity without energy dissipation.

In order to have surface wave in the context of linear theory of thermoelasticity without energy dissipation, proceeding on the same lines as in previous section and on simplification, we obtained

$$(1-\alpha_3^2)^2\left[\alpha_1^2 + \alpha_5^2 + \alpha_1\alpha_5 + 1 - \frac{c^2}{C_1^2}\right] + 4\alpha_1\alpha_5\alpha_3(\alpha_1 + \alpha_5) = 0, \tag{74}$$

and when $\varepsilon_1 = 0$, (74) reduces to,

$$\left(2 - \frac{c^2}{C_2^2}\right)^2 = 16\left(1 - \frac{c^2}{C_2^2}\right)\left(1 - \frac{c^2}{C_1^2}\right). \tag{75}$$

Here eqs. (74) and (75) are in GN theory, corresponds to the eqs. (52) and (53) of LS theory.

NUMERICAL RESULTS AND DISCUSSION

In this section, firstly, we found that characteristic eqs. (42)and (51) after lengthy calculations and reductions decoupled into symmetric and antisymmetric modes respectively.

$$\det(\Lambda'_s) = \det(\Lambda'_{1s})\det(\Lambda'_{2s}) = \sum_{k=1,3,5}(-1)^{\left(\frac{k+3}{2}\right)} r'_{33(k)} G'_k \tan^{\mp}(\gamma\alpha_k) = 0, \tag{76}$$

where $\gamma = \frac{\zeta d}{2}$

and

$$G'_1 = \begin{vmatrix} r'_{13(3)} & r'_{13(5)} \\ \Theta'_3 & \Theta'_5 \end{vmatrix}, \; G'_3 = \begin{vmatrix} r'_{13(1)} & r'_{13(5)} \\ \Theta'_1 & \Theta'_5 \end{vmatrix}, \; G'_5 = \begin{vmatrix} r'_{13(1)} & r'_{13(3)} \\ \Theta'_1 & \Theta'_3 \end{vmatrix} \tag{77}$$

of vibrations in both LS and GN theories of generalized thermoelasticity which are period equation.

Numerical calculations are then carried out to present phase and group velocities, (c and $U = c + \xi\frac{dc}{d\xi}$, respectively) dispersion curves Vs wave number, assuming the thickness of the plate, when the plate is free of stress. Dispersion curves for the first four symmetric and anti-symmetric modes are shown in Figure 1. and Figure 2. for LS theory and Figure 3. and Figure 4. for GN theory. The material chosen for this purpose of numerical evaluation is aluminum. The physical data for such materials is given as follows:

- *Young's modulus = 70 Gpa, Poisson ratio = 0.3, density= 2675 kg/m^3*
- *Specific heat =921 J/kg^o C ,thermal conductivity = 204 W / m^o C,*
- *Expansion coefficient = 23μ ε/^{o}C.*

In Figure 1(a), the first mode of symmetric vibration, the phase velocity decreases monotonically with increasing values of wave number from c_p (plate velocity) at ξ = 0 to c_R (Rayleigh surface wave speed) at $\xi = \infty$. The group velocity has the same asymptotic limits but has a minimum. In the first mode *antisymmetric vibration Figure 2(a), the phase velocity increases monotonically with increasing wave number values ξ from c = 0 at ξ = 0 to c = c_R at $\xi = \infty$. As $\xi \to 0$, $U \to 0$, which is characteristic of flexural waves, and as $\xi \to \infty$, $c \to U \to c_R$ in the plate. The maximum value of group velocity is equal to horizontal velocity of SV waves in the plate. The results obtained for flexural mode (first mode) are in agreement with the corresponding results obtained by Ewing et. al. [16](in Fig. 6-18) in the classical case.*

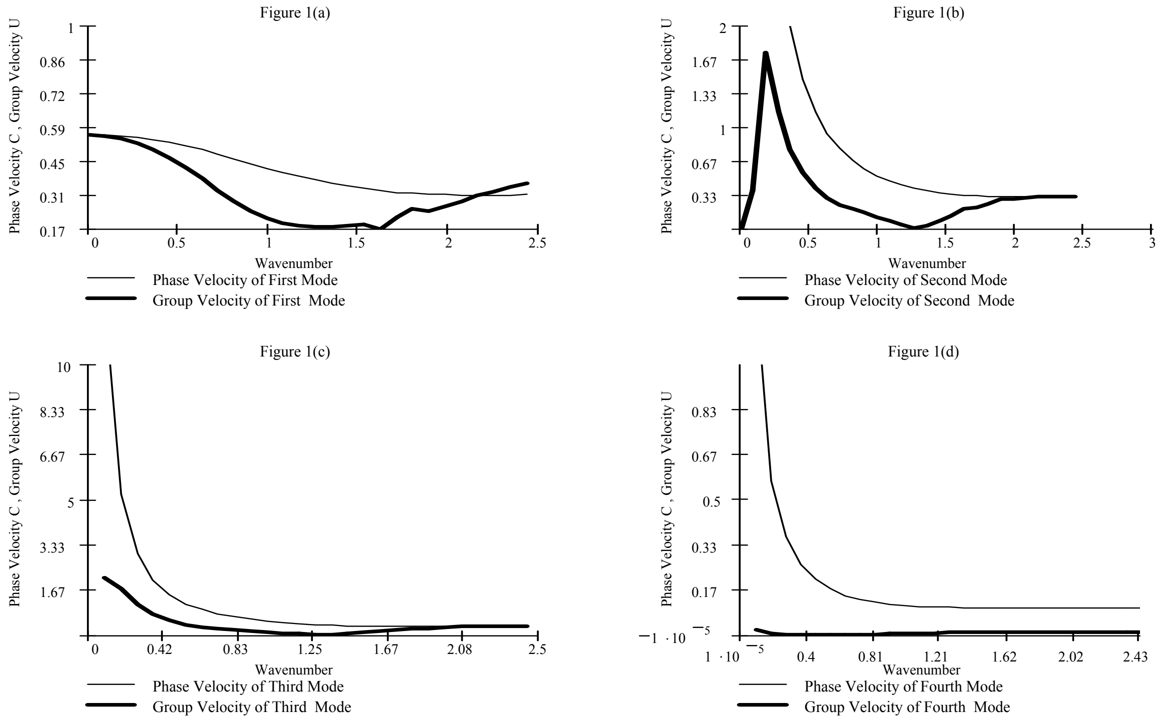

Figure 1. Dispersion curves in LS theory of generalized theories of thermoelasticity for symmetric modes

In Fig. 1(b), the second mode, the phase velocity is higher than the horizontal velocity of SV waves in the plate. Again as $c \to \infty$, $U \to 0$ as $\xi \to 0$ and as $\xi \to \infty$, $c \to U \to$ horizontal velocity of SV waves in the plate. Both the maximum and minimum values of group velocity are associated with this mode at intermediate wave numbers. Similar relations between phase and group velocity for higher modes are demonstrated in the dispersion curves in Figure 1(c), Figure 1(d) and Figure 2(c), Figure 2(d). The turning of the phase and group velocity curves for fourth mode (antisymmetric), Figure 2(d) approach the c-axis at low wave number, at such a large values that these are multiplied by 10^{-3} to see them on the figures.

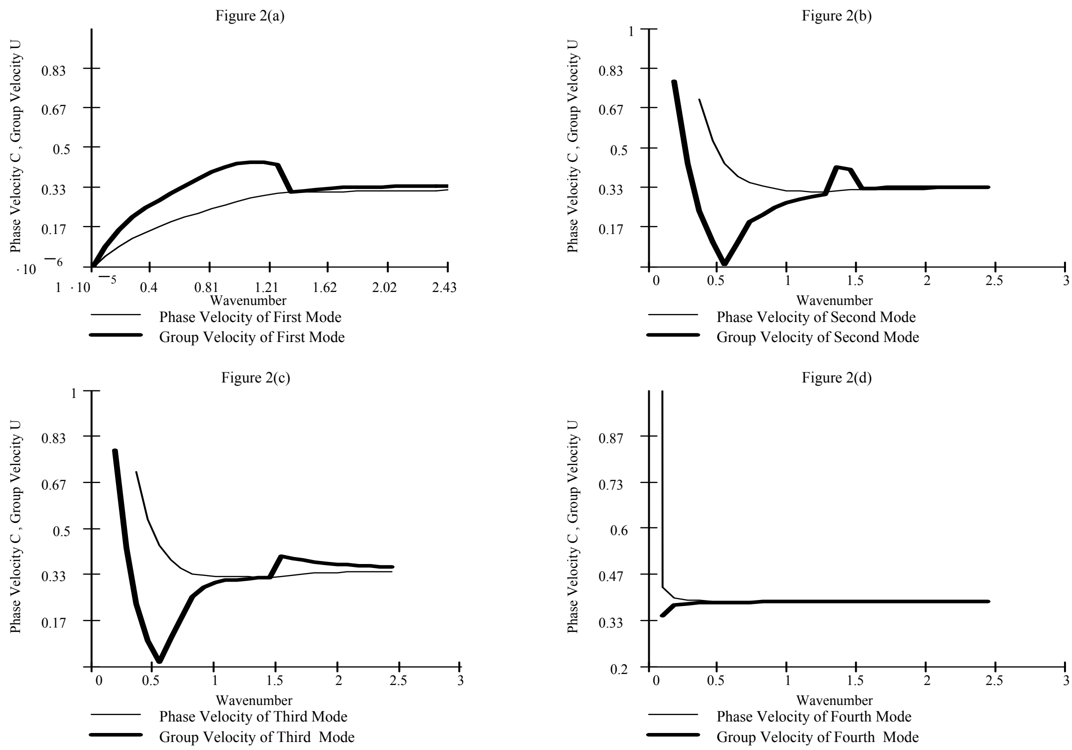

Figure 2. Dispersion curves in LS theory of generalized theories of thermoelasticity for antisymmetric modes

From Figure 1 and Figure 2, it is observed that dispersion curves for symmetric and antisymmetric modes in LS theory of generalized thermoelasticity get merged and then approach each other as wave number increases, where the phase and group velocities tend towards the Rayleigh surface wave speed.

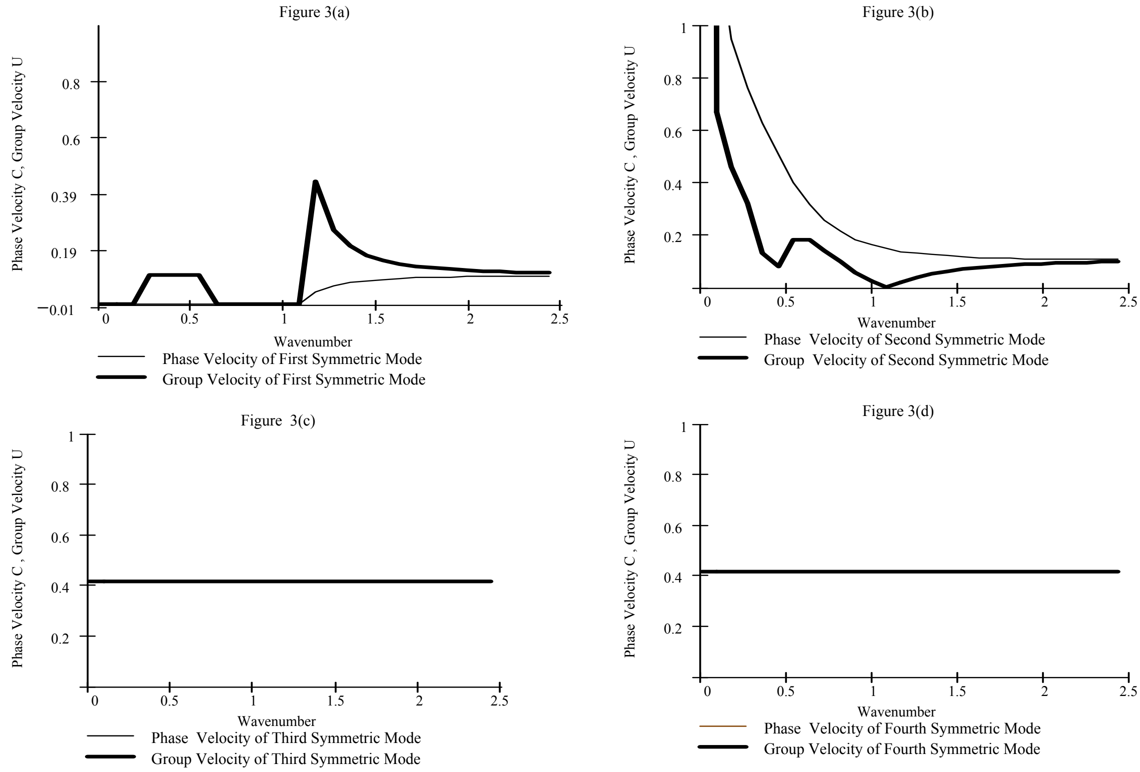

Figure 3. Dispersion curves in GN theory of generalized theories of thermoelasticity for symmetric modes

Dispersion curves for antisymmetric and symmetric modes in GN theory of generalized thermoelasticity, for aluminum material plate are shown in Figure 3 and Figure 4. It has been found from these Figure 3(c), Figure 4(c) and Figure 4(c), Figure 4(d) that phase velocity is equal to group velocity i.e., c = U for second and third modes (antisymmetric), third and fourth modes (symmetric), and hence these modes are non-dispersive in the GN theory.

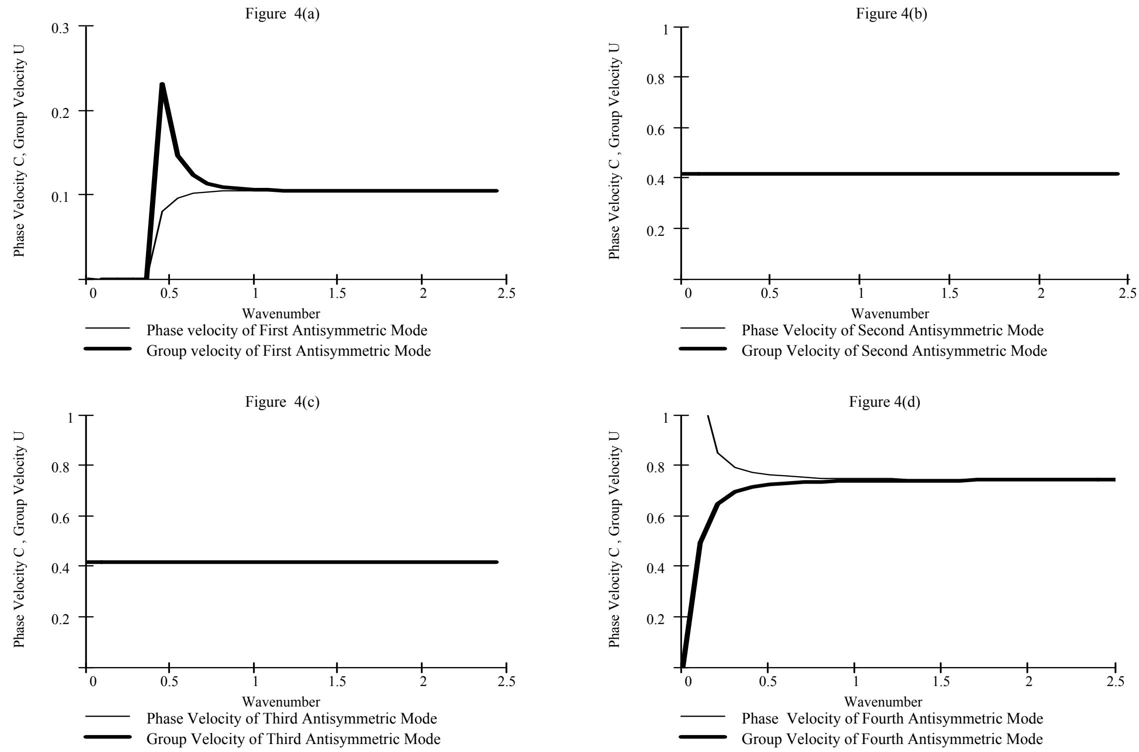

Figure 4. Dispersion curves in GN theory of generalized theories of thermoelasticity for antisymmetric modes

In the GN theory, Figure 1 and Figure 2, shows that there exist symmetric and antisymmetric modes of coupled (thermal and elastic waves modes) waves, without any attenuation. The fact that, *this is not the case in the LS theory is an interesting feature inherent in GN theory, in LS theory the waves experience attenuation, and the attenuation factors decay exponentially [21, 22]. It has also been observed that predictions of the GN theory are qualitatively similar to those of the LS theory.*

When the thermal relaxation time $\tau_0 \to 0$, then the results obtained in the analysis reduces to conventional coupled theory of thermoelasticity. When the coupling constant ε_1 is identically zero, the strain and thermal fields are uncoupled to each other. In this case the results can be obtained as in the uncoupled theory of thermoelasticity.

CONCLUSIONS

It is observed that implicitly there exist symmetric and antisymmetric modes of thermal and elastic waves modes (coupled) both in LS and GN theories of generalized thermoelasticity. It is found that in both the theories, waves mode are observed to be more effected at the zero wave number limits, due to the thermo-mechanical effects. This clearly demonstrates the difference between the coupled and generalized theory of thermoelasticity. The various waves mode get merged and then approach each others at high frequencies, where the phase velocity tends towards the Rayleigh surface wave speed. It observed that, although GN and LS models were derived from distinctively different physical assumptions and physical laws, the spectral behaviors described by GN model are qualitatively similar to that of LS model. They resolve two waves, one mechanical and one thermal wave, but in the GN model neither the mechanical wave nor the thermal wave experiences any attenuation. That, this is not the case in LS theory, is an interesting feature inherent in the GN theory, in LS theory both waves experience attenuation. It is also demonstrated that transverse waves are not functions of temperature field and thermo-mechanical terms in GN theory as observed in LS theory. Since, the classical uncoupled theory gives the identical results, therefore, it can be concluded that the differences between the generalized (GN and LS theories) and classical theories diminish in describing the phase velocities of uncoupled thermoelastic waves. When the wave number is assumed, the phase velocity can be obtained from eq. (42), consequently, the dispersion of phase velocity is given by eq. (42) when the plate is free of stress and using rigid boundary conditions. Further, for any c and ξ which satisfy eq. (42), the vector A1can be determined. Then A2 can be obtained by (36), and then the displacement, temperature as well as stress and strain, can be determined easily.

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