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[1.] Ajaib S. BANYAL

ON THERMAL CONVECTION IN MAGNETO – ROTATORY CONVECTION IN RIVLIN-ERICKSEN FLUID IN A POROUS MEDIUM

[1.] DEPARTMENT OF MATHEMATICS, GOVT. COLLEGE NADAUN (HAMIRPUR), 177033(HP), INDIA

ABSTRACT: The thermal instability of a couple-stress fluid acted upon by uniform vertical magnetic field and rotation heated from below is investigated. Following the linearized stability theory and normal mode analysis, the paper mathematically established the condition for characterizing the non-oscillatory motions which may be neutral or unstable for any combination of free and rigid boundaries at the top and bottom of the fluid. It is proved mathematically that all non-decaying slow motions starting from rest, in a couple-stress fluid of infinite horizontal extension and finite vertical depth, which is acted upon by uniform vertical magnetic field and rotation opposite to gravity and a constant vertical adverse temperature gradient, are necessarily non-oscillatory, in the regime established. The result is important since it hold for all wave numbers and for any combination of perfectly conducting dynamically free and rigid boundaries.
KEYWORDS: Thermal convection; Rivlin-Ericksen Fluid; Rotation; Rayleigh number; Taylor number
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INTRODUCTION

The thermal instability of a fluid layer with maintained adverse temperature gradient by heating the underside plays an important role in Geophysics, interiors of the Earth, Oceanography and Atmospheric Physics, and has been investigated by several authors (e.g., Bénard [4], Rayleigh [13], Jeffreys [8]) under different conditions. A detailed account of the theoretical and experimental study of the onset of Bénard Convection in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [6] in his celebrated monograph. The use of Boussinesq approximation has been made throughout, which states that the density changes are disregarded in all other terms in the equation of motion except the external force term. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been considered by Veronis [19].

The physics is quite similar in the stellar case, in that helium acts like in raising the density and in diffusing more slowly than heat. The condition under which convective motions are important in stellar atmospheres are usually far removed from consideration of single component fluid and rigid boundaries and therefore it is desirable to consider a fluid acted upon by a solute gradient with free or rigid boundaries. The problem is of great importance because of its applications to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering. Bhatia and Steiner [6] have considered the effect of uniform rotation on the thermal instability of a viscoelastic (Maxwell) fluid and found that rotation has a destabilizing influence in contrast to the stabilizing effect on Newtonian fluid.

Sharma [16] has studied the thermal instability of a layer of viscoelastic (Oldroydian) fluid acted upon by a uniform rotation and found that rotation has destabilizing as well as stabilizing effects under certain conditions in contrast to that of a Maxwell fluid where it has a destabilizing effect. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's [11] constitutive relations. Two such classes of fluids are Rivlin-Ericksen's and Walter's (model B') fluids. Rivlin-Ericksen [14] has proposed a theoretical model for such one class of elastico-viscous fluids. Sharma and kumar [17] have studied the effect of rotation on thermal instability in Rivlin-Ericksen elastico-viscous fluid and found that rotation has a stabilizing effect and introduces oscillatory modes in the system. Kumar et al. [9] considered effect of rotation and magnetic field on Rivlin-Ericksen elastico-viscous fluid and found that rotation has stabilizing effect; where as magnetic field has both stabilizing and destabilizing effects. A layer of such fluid heated from below or under *the action of magnetic field or rotation or both may find applications in geophysics, interior of the Earth, Oceanography, and the atmospheric physics. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable.*

In all above studies, the medium has been considered to be non-porous with free boundaries only, in general. In recent years, the investigation of flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. When a fluid permeates a porous material, the gross effect is represented by the Darcy's law. As a result of this macroscopic law, the usual viscous term in the equation of Rivlin-Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)q\right]$, *where* μ *and* μ' *are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid,* k_1 *is the medium permeability and* q *is the Darcian (filter) velocity of the fluid. The problem of thermosolutal convection in fluids in a porous medium is of great importance in geophysics, soil sciences, ground water hydrology and astrophysics. Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of the comets, meteorites and interplanetary dust strongly suggest the importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. Thermal convection in porous medium is also of interest in geophysical system, electrochemistry and metallurgy. A comprehensive review of the literature concerning thermal convection in a fluid-saturated porous medium may be found in the book by Nield and Bejan [10].*

Pellow and Southwell [12] proved the validity of PES for the classical Rayleigh-Bénard convection problem. Banerjee et al [2] gave a new scheme for combining the governing equations of thermohaline convection, which is shown to lead to the bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries and, Banerjee and Banerjee [1] established a criterion on characterization of non-oscillatory motions in hydrodynamics which was further extended by Gupta et al [7]. However no such result existed for non-Newtonian fluid configurations in general and in particular, for Rivlin-Ericksen viscoelastic fluid configurations. Banyal [3] have characterized the oscillatory motions in Rivlin-Ericksen fluid in the presence of magnetic field.

Keeping in mind the importance of non-Newtonian fluids, as stated above, this article attempts to study Rivlin-Ericksen viscoelastic fluid heated from below in the presence of uniform vertical magnetic field and uniform rotation in a porous medium, with rigid boundaries and it has been established that the onset of instability in a Rivlin-Ericksen viscoelastic fluid heated from below in a porous medium, in the presence of uniform vertical magnetic field and uniform rotation, cannot manifest itself as oscillatory motions of growing amplitude if Q *the Chandrasekhar number,* T_A *the Taylor number, F the viscoelasticity parameter,* ε *the porosity,* P_l *the medium permeability and* p_2 *the magnetic Prandtl number, satisfy the inequality,* $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \le 1$, *for all wave numbers and for any combination of perfectly conducting dynamically free and rigid boundaries.*

FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we Consider an infinite, horizontal, incompressible electrically conducting Rivlin-Ericksen viscoelastic fluid layer, of thickness d, heated from below so that, the temperature and density at the bottom surface z = 0 are T_0 *and* ρ_0, *and at the upper surface z = d are* T_d *and* ρ_d *respectively, and that a uniform adverse temperature gradient* $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ *is maintained. The gravity field* $\vec{g}(0,0,-g)$, *uniform vertical rotation* $\vec{\Omega}(0,0,\Omega)$ *and a uniform vertical magnetic field pervade on the system* $\vec{H}(0,0,H)$. *This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity* ε *and medium permeability* k_1.

Let p, ρ, *T,* α, g, η, μ_e *and* $\vec{q}(u,v,w)$ *denote respectively the fluid pressure, fluid density temperature, thermal coefficient of expansion, gravitational acceleration, resistivity, magnetic permeability and filter velocity of the fluid. Then the momentum balance, mass balance, and energy balance equation of Rivlin-Ericksen fluid and Maxwell's equations through porous medium, governing the flow of Rivlin-Ericksen fluid in the presence of uniform vertical magnetic field and uniform vertical rotation (Rivlin and Ericksen [14]; Chandrasekhar [6] and Sharma et al [18]) are given by*

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}\left(\vec{q}.\nabla\right)\vec{q}\right]=-\left(\frac{1}{\rho_0}\right)\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}+\frac{\mu_e}{4\pi\rho_o}(\nabla\times\vec{H})\times\vec{H}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right), \qquad (1)$$

$$\nabla.\vec{q}=0\,, \tag{2}$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T\,, \tag{3}$$

$$\varepsilon\frac{d\vec{H}}{dt}=(\vec{H}.\nabla)\vec{q}+\varepsilon\eta\nabla^2\vec{H}\,, \tag{4}$$

$$\nabla.\vec{H}=0\,, \tag{5}$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon^{-1}\vec{q}.\nabla$ *, stand for the convective derivatives. Here* $E=\varepsilon+(1-\varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_i}\right)$*, is a constant and while* ρ_s *,* c_s *and* ρ_0 *,* c_i *, stands for the density and heat capacity of the solid (porous matrix) material and the fluid, respectively,* ε *is the medium porosity and* $\vec{r}(x,y,z)$*.*

The equation of state is:

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right] \tag{6}$$

where the suffix zero refer to the values at the reference level z = 0. In writing the equation (1), we made use of the Boussinesq approximation, which states that the density variations are ignored in all terms in the equation of motion except the external force term. The kinematic viscosity ν *, kinematic viscoelasticity* ν' *, magnetic permeability* μ_e*, thermal diffusivity* κ*, and electrical resistivity* η*, and the coefficient of thermal expansion* α *are all assumed to be constants.*

The steady state solution is

$$\vec{q}=(0,0,0)\ ,\ \rho=\rho_0(1+\alpha\beta z)\,,\ T=-\beta z+T_0 \tag{7}$$

Here we use the linearized stability theory and the normal mode analysis method. Consider a small perturbations on the steady state solution, and let $\delta\rho$*,* δp*,* θ*,* $\vec{q}(u,v,w)$ *and* $\vec{h}=\left(h_x,h_y,h_z\right)$ *denote respectively the perturbations in density* ρ*, pressure p, temperature T, velocity* $\vec{q}(0,0,0)$ *and the magnetic field* $\vec{H}=(0,0,H)$*. The change in density* $\delta\rho$*, caused mainly by the perturbation* θ *in temperature is given by*

$$\delta\rho=-\rho_0(\alpha\theta)\,. \tag{8}$$

Then the linearized perturbation equations of the Rinlin-Ericksen fluid reduces to

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t}=-\frac{1}{\rho_0}(\nabla\delta p)-\vec{g}(\alpha\theta)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{h}\right)\times\vec{H}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right), \tag{9}$$

$$\nabla.\vec{q}=0\,, \tag{10}$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta\,, \tag{11}$$

$$\varepsilon\frac{\partial\vec{h}}{\partial t}=\left(\vec{H}.\nabla\right)\vec{q}+\varepsilon\eta\nabla^2\vec{h}\,. \tag{12}$$

and

$$\nabla.\vec{h}=0\,, \tag{13}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

NORMAL MODE ANALYSIS

Analyzing the disturbances into two-dimensional waves, and considering disturbances characterized by a particular wave number, we assume that the Perturbation quantities are of the form

$$\left[w,\theta,h_z,\varsigma,\xi\right]=\left[W(z),\Theta(z),K(z),Z(z),X(z)\right]\exp\left(ik_x x+ik_y y+nt\right), \tag{14}$$

where k_x, k_y *are the wave numbers along the x- and y-directions, respectively,* $k=\left(k_x^2+k_y^2\right)^{\frac{1}{2}}$*, is the resultant wave number, n is the growth rate which is, in general, a complex constant;* $\varsigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ *and*

$\xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$ denote the z-component of vorticity and current density respectively and $W(z), K(z), \Theta(z), \Gamma(z), Z(z)$ and $X(z)$ are the functions of z only.

Using (14), equations (9)-(13), within the framework of Boussinesq approximations, in the non-dimensional form transform to

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l}(1+\sigma F)\right](D^2 - a^2)W = -Ra^2\Theta - T_A DZ + Q(D^2 - a^2)DK, \tag{15}$$

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l}(1+\sigma F)\right]Z = DW + QDX, \tag{16}$$

$$(D^2 - a^2 - p_2\sigma)K = -DW, \tag{17}$$

$$(D^2 - a^2 - p_2\sigma)X = -DZ, \tag{18}$$

and

$$(D^2 - a^2 - Ep_1\sigma)\Theta = -W, \tag{19}$$

where we have introduced new coordinates (x', y', z') = (x/d, y/d, z/d) in new units of length d and $D = d/dz'$. For convenience, the dashes are dropped hereafter. Also we have substituted $a = kd, \sigma = \frac{nd^2}{\nu}$, $p_1 = \frac{\nu}{\kappa}$ is the thermal Prandtl number; $p_2 = \frac{\nu}{\eta}$ is the magnetic Prandtl number; $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability, $F = \frac{\nu'}{d^2}$ is the dimensionless viscoelasticity parameter of the Rivlin-Ericksen fluid; $R = \frac{g\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number; $Q = \frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon}$ is the Chandrasekhar number and $T_A = \frac{4\Omega^2 d^4}{\nu^2\varepsilon^2}$ is the Taylor number. Also we have substituted $W = W_\oplus, \Theta = \frac{\beta d^2}{\kappa}\Theta_\oplus, Z = \frac{2\Omega d}{\nu\varepsilon}Z_\oplus, K = \frac{Hd}{\varepsilon\eta}K_\oplus, X = \left(\frac{Hd}{\varepsilon\eta}\right)\left(\frac{2\Omega d}{\varepsilon\nu}\right)X_\oplus$ and $D_\oplus = dD$, and dropped $(\oplus)$ for convenience.

We now consider the cases where the boundaries are rigid-rigid or rigid-free or free-rigid or free-free at z = 0 and z = 1, as the case may be, and are perfectly conducting. The boundaries are maintained at constant temperature, thus the perturbations in the temperature are zero at the boundaries. The appropriate boundary conditions with respect to which equations (15) - (17), must possess a solution are:

W = 0 = Θ, on both the horizontal boundaries,
DW = 0=Z=K=DX, on a rigid boundary, or
$D^2W = 0$ =DZ=K=DX, on a dynamically free boundary, (20)

Equations (15)--(19), along with boundary conditions (20), pose an eigenvalue problem for σ and we wish to characterize σ_i, when $\sigma_r \geq 0$.

We first note that since W, K and Z satisfy $W(0) = 0 = W(1)$ and $K(0) = 0 = K(1)$ in addition to satisfying to governing equations and hence we have from the Rayleigh-Ritz inequality Schultz [15]

$$\int_0^1 |DW|^2 dz \geq \pi^2 \int_0^1 |W|^2 dz \text{ and} \int_0^1 |DK|^2 dz \geq \pi^2 \int_0^1 |K|^2 dz, \tag{21}$$

MATHEMATICAL ANALYSIS

We prove the following lemma:

Lemma 1: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 \left\{|DK|^2 + a^2|K|^2\right\}dz \leq \frac{1}{\pi^2}\int_0^1 |DW|^2 dz$$

Proof: Multiplying equation (17) by K (the complex conjugate of K), integrating by parts each term of the resulting equation on the left hand side for an appropriate number of times and making use of boundary conditions on K namely $K(0) = 0 = K(1)$, it follows that

$$\int_0^1 \left\{|DK|^2 + a^2|K|^2\right\}dz + \sigma_r p_2 \int_0^1 |K|^2 dz = \text{Real part of}\left\{\int_0^1 K^* DW dz\right\} \leq \left|\int_0^1 K^* DW dz\right| \leq \int_0^1 |K^* DW| dz,$$

$$\leq \int_0^1 |K^*||DW|dz \ \leq \int_0^1 |K||DW|dz \leq \left\{\int_0^1 |K|^2 dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}} \qquad (22)$$

(Utilizing Cauchy-Schwartz-inequality),

This gives that

$$\int_0^1 |DK|^2 dz \leq \left\{\int_0^1 |K|^2 dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}}, \qquad (23)$$

Inequality (22) on utilizing (23), gives

$$\left\{\int_0^1 |K|^2 dz\right\}^{\frac{1}{2}} \leq \frac{1}{\pi^2} \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}}, \qquad (24)$$

Since $\sigma_r \geq 0$ *and* $p_2 \rangle 0$, *hence inequality (22) on utilizing (24), give*

$$\int_0^1 \left(|DK|^2 + a^2|K|^2\right) dz \leq \frac{1}{\pi^2} \int_0^1 |DW|^2 dz, \qquad (25)$$

This completes the proof of lemma.

Lemma 2: *For any arbitrary oscillatory perturbation, neutral or unstable*

$$\int_0^1 |Z|^2 dz \leq P_l^2 \int_0^1 |DW|^2 dz \cdot$$

Proof: *Multiplying equation (16) by* Z^* *(the complex conjugate of Z), integrating by parts each term of the resulting equation on the left hand side for an appropriate number of times on utilizing equation (18) and appropriate boundary conditions (20), it follows that*

$$\left[\frac{\sigma_r}{\varepsilon} + \frac{1}{P_l}(1+\sigma_r F)\right] \int_0^1 |Z|^2 dz + Q\int_0^1 \left\{|DX|^2 + a^2|X|^2\right\} dz + Qp_2\sigma_r \int_0^1 |X|^2 dz$$

= *Real part of* $\left\{\int_0^1 DW^* Z dz\right\} \leq \left|\int_0^1 DW^* Z dz\right|$,

$$\leq \int_0^1 |DW^* Z| dz \leq \int_0^1 |DW^*||Z| dz,$$

$$= \int_0^1 |DW||Z| dz \leq \left\{\int_0^1 |Z|^2 dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}}, \qquad (26)$$

(Utilizing Cauchy-Schwartz-inequality),

This gives that $\frac{1}{P_l} \int_0^1 |Z|^2 dz \leq \left\{\int_0^1 |Z|^2 dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}}$,

Therefore, we get

$$\left\{\int_0^1 |Z|^2 dz\right\}^{\frac{1}{2}} \leq P_l \left\{\int_0^1 |DW|^2 dz\right\}^{\frac{1}{2}}, \qquad (27)$$

Since $\sigma_r \geq 0$ *and* $p_2 \rangle 0$, *utilizing (27), inequality (26), give*

$$\int_0^1 |Z|^2 dz \leq P_l^2 \int_0^1 |DW|^2 dz, \qquad (28)$$

This completes the proof of lemma.

We prove the following theorem:

Theorem 1: *If* $R \rangle 0$, $F \rangle 0$, $Q \rangle 0$, $T_A \rangle 0$, $P_l \rangle 0$, $p_1 \rangle 0$, $p_2 \rangle 0$, $\sigma_r \geq 0$ *and* $\sigma_i \neq 0$ *then the necessary condition for the existence of non-trivial solution* (W, Θ, K, Z, X) *of equations (15) - (19), together with boundary conditions (20) is that*

$$\left(\frac{\varepsilon P_l}{P_l + \varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right) + T_A P_l^2 \rangle 1 \cdot$$

Proof: *Multiplying equation (15) by* W^* *(the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get*

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1 W^*\left(D^2-a^2\right)Wdz=-Ra^2\int_0^1 W^*\Theta dz$$

$$-T_A\int_0^1 W^*DZdz+Q\int_0^1 W^*D\left(D^2-a^2\right)Kdz, \tag{29}$$

Taking complex conjugate on both sides of equation (19), we get

$$\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^*=-W^*, \tag{30}$$

Therefore, using (30), we get

$$\int_0^1 W^*\Theta dz=-\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz, \tag{31}$$

Also taking complex conjugate on both sides of equation (16), we get

$$\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]Z^*-QDX^*=DW^*, \tag{32}$$

Therefore, using (32), we get

$$\int_0^1 W^*DZdz=-\int_0^1 DW^*Zdz=-\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^*Zdz+Q\int_0^1 ZDX^*dz,$$

Integrating by parts the third term on left hand side and using equation (18), and appropriate boundary condition (20), we get

$$\int_0^1 W^*DZdz=-\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^*Zdz+Q\int_0^1 X\left(D^2-a^2-p_2\sigma\right)X^*dz, \tag{33}$$

Also taking complex conjugate on both sides of equation (17), we get

$$\left[D^2-a^2-p_2\sigma^*\right]K^*=-DW^*, \tag{34}$$

Therefore, equation (34), using appropriate boundary condition (20), we get

$$\int_0^1 W^*D\left(D^2-a^2\right)Kdz=-\int_0^1 DW^*\left(D^2-a^2\right)Kdz=\int_0^1 K\left(D^2-a^2\right)\left(D^2-a^2-p_2\sigma^*\right)K^*dz \tag{35}$$

Substituting (31), (33) and (35), in the right hand side of equation (29), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1 W^*\left(D^2-a^2\right)Wdz=Ra^2\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^*dz$$

$$+T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^*Zdz-T_AQ\int_0^1 X\left(D^2-a^2-p_2\sigma\right)X^*dz$$

$$+Q\int_0^1 K^*\left(D^2-a^2\right)^2Kdz-Qp_2\sigma^*\int_0^1 K^*\left(D^2-a^2\right)Kdz, \tag{36}$$

Integrating the terms on both sides of equation (36) for an appropriate number of times and making use of the appropriate boundary conditions (20), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz=Ra^2\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2+Ep_1\sigma^*|\Theta|^2\right)dz$$

$$-T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1|Z|^2dz-T_AQ\int_0^1\left(|DX|^2+a^2|X|^2+p_2\sigma|X|^2\right)dz$$

$$-Q\int_0^1\left(\left|D^2K\right|^2+2a^2|DK|^2+a^4|K|^2\right)dz-Qp_2\sigma^*\int_0^1\left(|DK|^2+a^2|K|^2\right)dz, \tag{37}$$

Now equating the imaginary parts on both sides of equation (37), and canceling $\sigma_i(\neq 0)$ *throughout, we get*

$$\left[\frac{1}{\varepsilon}+\frac{F}{P_l}\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz$$

$$=\left[-Ra^2Ep_1\int_0^1|\Theta|^2dz+T_A\left\{\frac{1}{\varepsilon}+\frac{F}{P_l}\right\}\int_0^1|Z|^2dz-T_AQp_2\int_0^1|X|^2dz+Qp_2\int_0^1\left(|DK|^2+a^2|K|^2\right)dz\right], \tag{38}$$

Now R $\rangle$ *0,* $Q\rangle 0$, $P_l\rangle 0$, $\varepsilon\rangle 0$ *and* $T_A\rangle$ *0, utilizing the inequalities (25) and (28), the equation (38) gives,*

$$\left[\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)-T_A P_l^2\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)-\left(\frac{Qp_2}{\pi^2}\right)\right]\int_0^1 |DW|^2 dz + I_1 \langle 0\,, \tag{39}$$

where $I_1 = a^2\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)\int_0^1 |W|^2 dz + Ra^2 Ep_1 \int_0^1 |\Theta|^2 dz + T_A Qp_2 \int_0^1 |X|^2 dz$,

Is positive definite, and therefore, we must have

$$\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \rangle 1\,. \tag{40}$$

Hence, if

$$\sigma_r \geq 0 \text{ and } \sigma_i \neq 0\text{, then } \left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \rangle 1\,. \tag{41}$$

And this completes the proof of the theorem.

Presented otherwise from the point of view of existence of instability as stationary convection, the above theorem can be put in the form as follow:-

Theorem 2: *The sufficient condition for the onset of instability as a non-oscillatory motions of non-growing amplitude in a couple-stress fluid in a porous medium heated from below, in the presence of uniform vertical magnetic field and rotation is that,* $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \leq 1$, *where* Q *is the Chandrasekhar number,* T_A *is the Taylor number, F is the viscoelasticity parameter,* ε *is the porosity,* P_l *is the medium permeability and* p_2 *is the magnetic Prandtl number, for all wave numbers and any combination of perfectly conducting dynamically free and rigid boundaries.*

or

The onset of instability in a couple-stress fluid in a porous medium heated from below, in the presence of uniform vertical magnetic field and rotation, cannot manifest itself as oscillatory motions of growing amplitude if the Taylor number T_A, *the Chandrasekhar number Q, the porosity* ε, *the medium permeability* P_l, *the magnetic Prandtl number* p_2 *and the couple-stress parameter F, satisfy the inequality* $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \leq 1$, *for all wave numbers and any combination of perfectly conducting dynamically free and rigid boundaries.*

The sufficient condition for the validity of the 'PES', can be expressed in the form:

Theorem 3: *If* (W,Θ,K,Z,X,σ), $\sigma=\sigma_r+i\sigma_i$, $\sigma_r \geq 0$ *is a solution of equations (15) - (19), with R* $\rangle$ *0 and,*

$$\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \leq 1\,,$$

Then $\sigma_i = 0$.

In particular, the sufficient condition for the validity of the 'exchange principle' i.e.,

$$\sigma_r = 0 \Rightarrow \sigma_i = 0 \text{ if} \left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \leq 1\,.$$

In the context of existence of instability in 'oscillatory modes' and that of 'overstability' in the present configuration, we can state the above theorem as follow:-

Theorem 4: *The necessary condition for the existence of instability in 'oscillatory modes' and that of 'overstability' in a couple-stress fluid in a porous medium heated from below, in the presence of uniform vertical magnetic field and rotation is that the Taylor number* T_A, *the Chandrasekhar number Q, the porosity* ε, *the magnetic Prandtl number* p_2, *the couple-stress parameter of the fluid F and the medium permeability* P_l, *must satisfy the inequality* $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2 \rangle 1$, *for all wave numbers and any combination of perfectly conducting dynamically free and rigid boundaries.*

Special Cases: *It follows from theorem 1 that an arbitrary neutral or unstable mode is non-oscillatory in character and 'PES' is valid for:*

(i). Thermal convection in couple-stress fluid heated from below i. e. when Q = 0 = T_A. *(Sharma et al, 2001).*

(ii). Magneto-thermal convection in couple-stress fluid heated from below (T_A=0), if $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)\le 1$,

(iii). Rotatory-thermal convection in couple-stress fluid heated from below (Q = 0), if $T_A \le \frac{1}{P_l^2}$.

CONCLUSIONS

This theorem mathematically established that the onset of instability in a couple-stress fluid in the presence of uniform vertical rotation, cannot manifest itself as oscillatory motions of growing amplitude if the Taylor number T_A, the Chandrasekhar number Q, the porosity ε, the magnetic Prandtl number p_2, the couple-stress parameter of the fluid F and the medium permeability P_l, satisfy the inequality $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2\le 1$, for all wave numbers and any combination of perfectly conducting dynamically free and rigid boundaries.

The essential content of the theorem, from the point of view of linear stability theory is that for the configuration of couple-stress fluid of infinite horizontal extension heated form below, for any combination of perfectly conducting dynamically free and rigid boundaries. In the presence of uniform vertical magnetic field and rotation, parallel to the force field of gravity, an arbitrary neutral or unstable modes of the system are definitely non-oscillatory in character if $\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\left(\frac{Qp_2}{\pi^2}\right)+T_A P_l^2\le 1$, and in particular PES is valid.

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