ANNALS OF FACULTY ENGINEERING HUNEDOARA – International Journal of Engineering
Tome XI (Year 2013) – FASCICULE 4 (ISSN 1584 – 2673)

[1.] Adalbert KOVÁCS

ON THE STUDY OF THE FLUID'S NON-STATIONARY MOTION THROUGH NETWORK PROFILES USING THE BEM METHOD

[1.] UNIVERSITY "POLITEHNICA" OF TIMISOARA, DEPARTMENT OF MATHEMATICS, ROMANIA

ABSTRACT: This paper presents recent developments in using the CVBEM method for the study of the incompressible fluid's non-stationary motion through a network of profile grids, where the fluid's non-stationary motion is caused by small vibrations of the blades. Based on the theory of linearizability, the non-stationary motion is decomposed into a basic stationary motion and a non-stationary motion resulting from the vibrations of the blades with small amplitudes. Using the fundamental integral equation of the non-stationary motions's complex conjugate velocity, we establish the following two transfer kernels: $G(z,\zeta)$ describing stationary effects and $H(z,\zeta)$ describing non-stationary effects. We argue that the integral equation can be solved using the indirect BEM method of the non-stationary motion.
KEYWORDS: Incompressible fluid, non-stationary motion, profile grids, complex velocity, theory of linearizability, boundary element method

INTRODUCTION

The fluid's motion through turbomachines is in general non-stationary due to the vibration of hydraulic machinery blades or to the influence of the fluid viscosity. Indeed, the workflow in a turbomachine can be regarded stationary only in the sense that it is repeated cyclically over a full rotation of the rotor. However, according to [10], even this basic cycle is disrupted by various non-stationary phenomena for the following reasons. (a) The network layer is formed by a contour whose radius varies along the flow. These variations lead to essential irregularities in velocity and pressure. (b) Further, as the stream's structure is determined also by the viscosity, the boundary layer at infinity yields irregularities in the velocity around the network profiles. (c) Finally, the non-stationary components of the uid's motion are caused also by the oscillations (vibrations) of the network profiles.

The perturbations introduced by these factors are manifested by the existence of vortices in the network profiles. These perturbations are practically transmitted in the profile traces and are preserved only on finite distances, after which they are amortized. Even though the attenuation of the vortex intensity is asymptotic, from the mathematical point of view it is safe to consider that the layers of free vortices resulting from high intensity profiles are of finite length - see e.g. [3], [4], [9], [10], [11], [7].

Following this observation, in this paper we study the incompressible fluid's non-stationary motion through a network of profile grids, where the fluid's non-stationary motion is caused by small vibrations of the blades. For doing so, we use the theory of linearizability in conjunction with the boundary element method (BEM), as follows. Based on the theory of linearizability, we split the fluid's non-stationary motion in two parts: a basic stationary motion (Section 3) and a non-stationary motion resulting from the vibrations of the blades with small amplitudes (Section 4). Based on [3, 4], we establish the fundamental integral equation of the non-stationary motion's velocity and apply the BEM method to solve the non-stationary part of the fluid's motion (Section 5). To this end, we analyze the free vortices around the network profiles and consider the fluid's motion through the network profiles variable over time.

PRELIMINARIES

We consider the motion of an ideal and incompressible uid through an infinite network of profiles in the complex plane $z = x + iy$, with periodicity $\omega = te^{i\frac{\pi}{2}}$. We assume a constant average stream at infinity upstream and downstream, and denote respectively by $\overline{V}_{1\infty}$ and $\overline{V}_{2\infty}$ the upstream and downstream velocities. Further, the oscillations (vibrations) of network profiles are considered to *be synchronous harmonic, with a frequency f and a phase angle α from profile to profile. The vibrations' amplitude is assumed to be small. Finally, we consider a planar motion relative to the xOy Cartesian coordinate system, where the Ox axis is oriented in the flow direction such that it is perpendicular to the network's director lane.*

Definition 2.1 *The complex coordinates z_k are called the congruent points of the network profiles if they satisfy the following condition:*

$$z_k = z_0 + ikt + \alpha f(z) e^{j(f\tau - k\alpha)}, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{1}$$

where τ denotes the time, t is the network's step, k is the number of network profiles, a is the vibrations' amplitude, the function $f(z) = f_x(x, y) + i f_y(x, y)$ defines the shape of vibrations, and j denotes the imaginary unit linked to the time periodicity of processes that do not interact with the imaginary unit i.

Using the theory of linearizability, we conclude that the non-stationary motion through profile grids can be studied as the composition of a basic stationary motion and a non-stationary motion of perturbations. We thus have:

$$\overline{V}(z) = \overline{w}_0(z) + \overline{w}(z); \tag{2}$$

where $\overline{V}(z)$ denotes the non-stationary motion's complex conjugate velocity, $\overline{w}_0(z)$ is the stationary motion's complex conjugate velocity, and $\overline{w}(z)$ is the complex conjugate velocity of the non-stationary motion of perturbations. Using Eq. (2), in order to determine the non-stationary motion's complex conjugate velocity $\overline{V}(z)$, we thus need to establish and solve the integral equations of $\overline{w}_0(z)$ (see Section 3) and $\overline{w}(z)$ (see Section 4). To this end, we make the following considerations.

Proposition 2.2 *The complex conjugate velocity $\overline{w}(z)$ of the non-stationary motion of perturbations satisfies the following conditions:*

- $\overline{w}(z)$ *is periodic w.r.t. the time τ and step t of the network, and we thus have:*

$$\overline{w}(z + ikt, \tau) = \overline{w}(z) e^{j(f\tau - k\alpha)}, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{3}$$

- $\overline{w}(z \to 0)$ *when $x \to \pm\infty$, that is, the non-stationary motion of perturbations disappears at infinity, before and after the network.*

Due to the non-stationary nature of the fluid's motion, the circulation around each profile depends on time. However, this time dependency contradicts Thompson's theorem according to which the circulation along a closed curve (e.g. along the base profile L_0) is constant if the acceleration potential is uniform - see [3], [10]. For this reason, it is necessary to assume the existence of velocity discontinuities after each profile - see Fig. 1. These discontinuities are in fact free vortex layers which continually emerge from the top of each profile. Moreover, under the assumption that the fluid is incompressible and continuous, these discontinuities modify only the tangential components w_τ of the $\overline{w}(z)$ velocity, and the normal components w_n of the non-stationary motion of perturbations remain continuous. Similarly, the pressure fields are continuous and we conclude the following property.

Proposition 2.3 *The intensity $\eta_k(x, \tau)$ of free vortex layers is defined as*

$$\eta_k = w_{\tau^+}^{(k)} - w_{\tau^-}^{(k)} = 2 w_\tau^{(k)}, \quad w_{\tau^+}^{(k)} = -w_{\tau^+}^{(k)} = w_\tau^{(k)} \tag{4}$$

where $w_{\tau^+}^{(k)}$, $w_{\tau^-}^{(k)}$ respectively denote the tangential velocities of the discontinuities along the profile L_k.

When compared to the overall velocity, the non-stationary motion occurs with relatively low velocities. We thus assume that the asymptotic downstream velocity of vortex layers is $V_{2\infty} e^{-i\beta_2}$ - see Fig. 1(b). Using [10], we also assume that the vortex traces are rectilinear, with high intensity and of finite length.

Proposition 2.4 *[3], [10] The intensity $\eta_k(x, \tau)$ of free vortices is determined by the bound vortices $\gamma_k(x, \tau)$, that is by the circulation around the profiles, as follows:*

$$\eta_k(x, \tau) = 2 w_\tau = -\frac{1}{V_{2\infty}} \left(\frac{d\Gamma_k}{d\tau} \right)_{\tau = \tau_+} \tag{5}$$

where $\tau - \tau_+ = \dfrac{x - x_0}{V_{2\infty} \cos \beta_2}$ and $\Gamma_k = \Gamma_{0k} \cdot e^{-if\tau}$ denotes the circulation's stationary part around L_k.

Using Proposition 2.4, in Section 4 we will build a single integral equation for computing the velocity of the non-stationary motion of perturbations. We also show that the derived integral equation yields both necessary and sufficient conditions in the study of fluid's motion. When compared *to the work of [9], our approach brings an important benefit for the following reason. Instead of one equation, in [9] two integral equations are computed and studied: one Voltera equation to derive the vortex intensity and one equation to compute the induced velocity of perturbations.*

INTEGRAL EQUATION OF THE STATIONARY MOTION'S VELOCITY

In [6] we showed that the hydrodynamics of network profiles admits precisely four boundary problems, as listed below.

- ☐ ***Problem 1 (P1)**: The motion of an incompressible fluid through profile grids, where the complex potential is a holomorphic function and the domain is infinite-connex;*
- ☐ ***Problem 2 (P2)**: The motion of a compressible fluid through an isolated profile, where the complex potential is not a holomorphic function and the domain is simple-connex;*
- ☐ ***Problem 3 (P3)**: The motion of a compressible fluid through profile grids, where the complex potential is not a holomorphic function and the domain is infinite-connex;*
- ☐ ***Problem 4 (P4)**: The motion of a compressible fluid through profile grids on an axial-symmetric flow-surface, in variable thickness of stratum, where the complex potential is not a holomorphic function and the domain is infinite-connex.*

The above listed problems can be solved using BEM - see [6]. Moreover, solving problem P4 yields also solutions for problems P1, P2, and P3. That is, by appropriately adjusting the fluid's density ρ and stratum thickness h, problems P1, P2, and P3 become special cases of P4.

For computing the stationary motion's velocity $\overline{w}_0(z)$ in Eq. (2), in this paper we are interested in solving P1. As argued in [6], when both ρ and h are constant, then P4 becomes P1. Using the theory of p-analytic functions from [8] and the Cauchy integral equation of p-analytic functions, the integral equation of the complex conjugate velocity $\overline{w}_0(z)$ is derived in [4], [5] as given below:

$$\overline{w}_0(z) = \overline{V}_m + \int_{L_0} G(z,\zeta)\overline{w}(\zeta)d\zeta\,, \quad G(z,\zeta) = \frac{1}{2\pi i}\operatorname{ctg}\frac{\pi}{t}(z-\zeta) \tag{6}$$

where $\overline{V}_m = \dfrac{\overline{V}_{1\infty} + \overline{V}_{2\infty}}{2}$ is the asymptotic mean velocity and $G(z,\zeta)$ defines the kernel of the stationary motion. Let $\gamma(s)$ denote the intensity of bound vortices and $q(s)$ the intensity of profile sources. Using the hydrodynamic relation $\overline{w}(\zeta)d\zeta = (\gamma(s) + iq(s))ds$, from Eq. (6) we get:

$$\overline{w}_0(z) = \overline{V}_m + \int_{L_0} G(z,\zeta)(\gamma(s) + iq(s))ds \tag{7}$$

and conclude the following theorem.

***Theorem 3.1** In the case of the incompressible fluid's stationary motion, the stationary complex velocity $\overline{w}_0(z)$ in point $z \in D^-$ results from the composition of the following two complex velocities:*

- ☐ *the stationary velocity of the asymptotic motion, determined by $\overline{V}_m$;*
- ☐ *the stationary velocity resulting from the sources $\gamma(s) + iq(s)$ along the profile L_0.*

We note that Theorem 3.1 allows one to solve problem P1 by using the BEM method. To this end, in [5] a calculus algorithm for deriving the fluid's stationary velocity was given.

INTEGRAL EQUATION OF THE NON-STATIONARY MOTION'S VELOCITY

Let C_k; $k \in Z$ denote the closed curves around the profiles and their free vortex layers. In what follows, L_k, $k \in Z$ denote the profile contours, L_k^* the free vortex lines, and D_k^+ and D_k^+ the internal and, respectively, the external domain of the profiles L_k situated in a periodical strip with width t - see Fig. 1(a). Due to the periodic nature of the fluid's motion, it is sufficient to study the fluid's motion in the external domain D_0 of the principal periodical strip containing the base profile L_0 -see Fig. 1(b).

We make the following observations over Fig. 1(a). The contour C, enclosing the field point $z \in D^-$, is composed by slices and

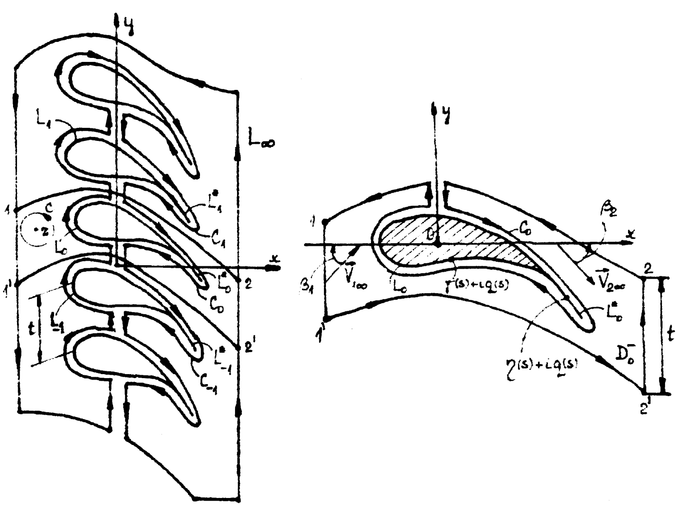

Figure 1: (a) Velocity discontinuities on profiles. (b) Velocity discontinuities on L_0.

we have: $C = L_\infty - \sum_{k=-\infty}^{\infty} C_k$ *Using the Cauchy integral equation for multiple-connex domains, the following equation then defines the perturbation's complex velocity w(z):*

$$\overline{w}(z) = \frac{1}{2\pi i}\int_{L_\infty} \frac{\overline{w}(\zeta)d\zeta}{\zeta - z} - \sum_{k=-\infty}^{\infty} \frac{1}{2\pi i}\int_{C_k} \frac{\overline{w}(\zeta_k)d\zeta_k}{\zeta_k - z} \tag{8}$$

By Proposition 2.2, we have $\overline{w}(z) \to 0$ *when* $x \to \pm\infty$. *That is, the complex velocity of the perturbations' non-stationary motion disappears at infinity, before and after the network. Based on Liouville's theorem, we conclude that the integral entity over* L_∞ *in Eq. (8) becomes constant, and thus zero. We further note the motion's periodicity w.r.t. the network's step t. From Proposition 2.2, we have* $\overline{w}(\zeta + ikt) = \overline{w}(\zeta)e^{-jk\alpha}$. *Based on these observations, Eq. (8) becomes as follows:*

$$\overline{w}(z) = \frac{1}{2\pi i}\oint_{C_0}\left(\sum_{k=\infty}^{\infty} \frac{\overline{w}(\zeta)e^{-jk\alpha}}{z - \zeta - ikt}\right)d\zeta = \frac{1}{2\pi i}\oint_{C_0}\left(\frac{1}{z-\zeta} + \sum_{k=1}^{\infty}\left(\frac{e^{jk\alpha}}{z-\zeta+ikt} + \frac{e^{-jk\alpha}}{z-\zeta-ikt}\right)\right)\overline{w}(\zeta)d\zeta \tag{9}$$

From the Euler relations, we have $e^{\pm jk\alpha} = \cos k\alpha \pm j\sin k\alpha$. *Using the series representation of the cotangent function, we obtain* $\mathrm{ctg} z = \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{1}{z - m\pi} + \frac{1}{z + k\pi}\right)$ *and* $\mathrm{ctg} z = i\,\mathrm{cth}\, iz$ *According to [10], Eq. (9) then becomes:*

$$\frac{1}{z-\zeta} + \sum_{k=1}^{\infty}\left(\frac{e^{jk\alpha}}{z-\zeta+ikt} + \frac{e^{-jk\alpha}}{z-\zeta-ikt}\right) = \frac{\pi}{t}\left(\frac{\mathrm{ch}\frac{(\pi-\alpha)(z-\zeta)}{t}}{\mathrm{sh}\frac{\pi(z-\zeta)}{t}} - ij\frac{\mathrm{sh}\frac{(\pi-\alpha)(z-\zeta)}{t}}{\mathrm{sh}\frac{\pi(z-\zeta)}{t}}\right) \tag{10}$$

By using Eq. (10) in conjunction with the curvilinear integral properties on $C_0 = L_0 \cup L_0^*$ *and applying the indirect BEM method to the integral equation Eq. (9) of the perturbation's complex velocity, we derive the following relation:*

$$\overline{w}(z) = \oint_{L_0} H(z,\zeta,\alpha)\overline{w}(\zeta)d\zeta + \int_{L_0^*} H(z,\zeta,\alpha)\overline{w}(\zeta)d\zeta \tag{11}$$

where $H(z,\zeta,\alpha) = \frac{1}{2ti}\left[\frac{ch\frac{(\pi-\alpha)(z-\zeta)}{t}}{sh\frac{\pi(z-\zeta)}{t}} - ij\frac{sh\frac{(\pi-\alpha)(z-\zeta)}{t}}{sh\frac{\pi(z-\zeta)}{t}}\right]$. *Consider now a point* $\zeta \in L_0$ *The perturbation's complex velocity in* ζ *is then:*

$$\overline{w}(\zeta) = \overline{v}(\zeta) + \overline{v}_0(\zeta) \tag{12}$$

where $\overline{v}(\zeta)$ *is the perturbation's velocity on the contour, that is the perturbation's relative velocity; and* $\overline{v}_0(z)$ *is the velocity resulting from the profile oscillations, that is the perturbation's transfer velocity. By replacing Eq. (7), Eq. (11) and Eq. (12) in Eq. (2), the fundamental integral equation corresponding to the complex velocity* $\overline{V}(z)$ *of fluid's non-stationary motion is given below:*

$$\begin{aligned}\overline{V}(z) = \overline{V}_m &+ \int_{L_0} G(z,\zeta)(\gamma(s) + iq(s))ds(\zeta) + \int_{L_0} H(z,\zeta,\alpha)(\gamma(s)+iq(s))ds(\zeta) + \\ &+ \int_{L_0^*} H(z,\zeta,\alpha)(\eta(s)+iq(s))ds(\zeta) + \int_{L_0} H(z,\zeta,\alpha)\overline{v}_0(\zeta)d\zeta\end{aligned} \tag{13}$$

where $\gamma(s)+iq(s)$ *represents the bound sources intensity on* L_0, *and* $\eta(s)+iq(s)$ *the sources intensity on the discontinuity lane* L_0^*. *Following Eq. (13), we derive the following theorem.*

Theorem 4.1 *In the case of the incompressible fluid's non-stationary motion, the fluid's complex velocity* $\overline{V}(z)$ *in point* $z \in D^-$ *results from the composition of the following five complex velocities:*

- *the stationary velocity of the asymptotic motion, determined by* $\overline{V}_m$;
- *the stationary velocity resulting from the sources* $\gamma(s)+iq(s)$ *along* L_0;
- *the perturbation velocity resulting from the sources* $\gamma(s)+iq(s)$ *along* L_0;
- *the perturbation velocity resulting from the sources* $\eta(s)+iq(s)$ *along the discontinuity lane* L_0^* *of the free vortices* $\eta(s)$;

- the transfer velocity resulting from the oscillations of L_0, determined by the velocity $\overline{v}_0(\zeta)$.

From Theorem 4.1, we conclude that the fluid's non-stationary motion is determined by the following two kernels: $G(z,\zeta)$ defining the transfer kernel of stationary effects, and $H(z,\zeta,\alpha)$ characterizing the transfer kernel of the perturbations.

SOLVING THE INTEGRAL EQUATION OF THE PERTURBATION'S COMPLEX VELOCITY

Following the results of Section 4, we now discuss considerations on solving the integral equation of the perturbation's complex velocity $\overline{w}(z)$.

We start with the following observation. Under the assumption that the profile vibrations are rectilinear harmonic vibrations along the Oy axis, we have the following identity:

$$\overline{v}_0(z) = jf(y)\cdot e^{jf\tau}\cdot e^{-i\beta_0} \tag{14}$$

where β_0 is the angle formed by the vector $\vec{v}_0$ with the Ox axis.

In order to satisfy the boundary conditions of Eq. (10) from the CVBEM method, we consider Eq. (11) on the boundary, that is on $z \to z_0 \in (L_0 \cup L_0^*)$. To this end, we make use of the Plemelj formulas in the first integral of the right-hand side of Eq. (11), that is the following identity is applied:

$$\lim_{z\to z_0}\int_{L_0} H(z,\zeta,\alpha)\overline{w}(\zeta)d\zeta = {\int_{L_0}}' H(z,\zeta,\alpha)\overline{w}(\zeta)d\zeta - \frac{1}{2}\overline{w}(z_0) \tag{15}$$

where the primed integral notation ${\int_{L_0}}'$ denotes the fact that the integral expression is calculated w.r.t. the direction of the principal values.

Based on the results of [3], we then conclude the property given below.

Proposition 5.1 In the points $\zeta \in L_0^*$, the complex velocity $\overline{w}_n(s)$ satisfies the following relation:

$$\int_{L_0^*} H(z,\zeta,\alpha)\overline{w}_n(\zeta)d\zeta = 0 \tag{16}$$

Using Eq. (12) and Eq. (15) in conjunction with Proposition 5.1 for $z \to z_0 \in (L_0)$, the integral equation Eq. (11) can be written in the following form:

$$\overline{v}(z_0) = 2{\int_{L_0}}' H(z_0,\zeta,\alpha)\overline{v}(\zeta)d\zeta + 2\int_{L_0^*} H(z_0,\zeta,\alpha)\overline{w}(\zeta)d\zeta + 2{\int_{L_0}}' H(z_0,\zeta,\alpha)\overline{v}_0(\zeta)d\zeta - \overline{v}_0(z_0) \tag{17}$$

where $\overline{w}(\zeta)$ can be expressed as a function of the circulation around the base profile L_0, that is as a function of $\overline{v}(\zeta)$, using Eq. (5).

As a result, the integral equation Eq. (17) contains only one unknown function, that is $\overline{v}(\zeta)$. The values of $\overline{v}(\zeta)$ can be computed by applying the discretization procedure of the CVBEM method. By using the values of $\overline{v}(\zeta)$ for each $\zeta \in L_0 \cup L_0^*$, the complex velocity $\overline{V}(z)$ of the fluid's non-stationary motion can further be derived from Eq. (13) in every point z $z \in D_0^-$.

We finally make the following observation. Let us assume that the profile vibrations are rectilinear harmonic vibrations along the Oy axis as given in Eq. (14). Further, instead of the CVBEM method, let us apply the BEM method with real values. Under these assumptions, following the results of [11], the real part of the integral equation Eq. (17) can be split and a second-order Fredholme integral equation is derived for $V(\theta)$, with $\theta \in [0,2\pi]$.

CONCLUSIONS

Using the theory of linearizability, we showed that the fluid's non-stationary motion through a network of profile grids is determined by a basic stationary motion and a non-stationary perturbation motion resulting from the vibrations of the blades and the fluid's viscosity. Using the BEM method, the fundamental integral equation of the fluid's non-stationary motion was derived and solved. We argued that the fluid's motion is characterized by the transfer kernel $G(z,\zeta)$ of stationary effects in conjunction with the transfer kernel $H(z,\zeta,\alpha)$ of perturbations.

REFERENCES

[1.] Banerjee, P. K. and Butter_eld, R. (1984). Boundary Element Methods in Engineering Sciences (in Russian). Mir, Moscow.

[2.] Hromadka, T. V. II. (1984). The Complex Variable Boundary Element Method. Springer-Verlag, Berlin.

[3.] Kovács, A. (2005). Classical and Modern Mathematical Methods in the Hydrodynamics of Profile Grids (in German). Ed. Politehnica Timisoara, Romania.

[4.] Kovács, A. (1997). On the Definition and the Solution of the Integral Equation of the Complex Velocity of the Non-Stationary Motion through Profile Grids (in German). In Proc. of the 7th Symposium of Mathematics and its Applications, number 6/9.XI.1997, pages 129-134, Timisoara, Romania.

[5.] Kovács, A. and Kovács, L. (2003). The Calculus Algorithm for Integral Equation of the Compressible Fluids Velocity Potential through Profile Grids (in German). In Proc. of the 10th Symposium of Mathematics and its Applications, number 6/9.XI.2003, pages 427-434, Timisoara, Romania.

[6.] Kovács, A. and Kovács, L. (2009). Solving the Four Fundamental Problems from the Theory of Profile Grids via BEM. In Proc. of the 12th Symposium of Mathematics and its Applications, number 5/7XI.2009, pages 393-400, Timisoara, Romania.

[7.] Kovács, A. and Kovács, L. and Kovács, L. (2012). The Boundary Element Method in the Study of the Fluid's Non-Stationary Motion Through Network Profiles. In Proc. of the 13th Symposium of Mathematics and its Applications, number ¼ XI.2012, pages 241-248, Timisoara, Romania.

[8.] Polojii, G. N. (1973). The Theory and Applications of p-Analytic and (p,q)-Analytic Functions (in Russian). Nauka Dumka, Kiev.

[9.] Popescu, I. L. (1967). Non-Permanent Motions in the Hydrodynamic Plane (in Romanian). Ed. Did. Ped. Bucuresti, Romania.

[10.] Samoilovitsch, G. S. (1975). The Perturbation of the Blades' Oscillation in Turbomachines (in Russian). Masinostroenie, Moscow.

[11.] Samoilovitsch, G. S., Kasimirski, Z. B., and Nitusov, V. V. (1968). The Calculus of the Network motion with Vibrations through an Incompressible Fluid Flow. MJG Izv. ANSSSR, (5):47-53.

ANNALS of Faculty Engineering Hunedoara

– International Journal of Engineering