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CALCULATION OF THE LOAD CAPACITY OF GEAR COUPLING BASED ON THE CONTACT STRESS

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Abstract: The gear couplings can compensate misalignments between two connected shafts. Main components of gear coupling are the hub and the sleeve. They create a special gearing having intersecting axes, when angular misalignment is occurred. Because of the crowned tooth surface of the hub the meshing is point contact in any instant. In this paper the position of contact points and the curvatures are determined using mathematical models of tooth surfaces. The load carrying capacity of gear coupling is restricted by the contact stress, which will be determined based on the Hertz theory.
Keywords: gear coupling, point contact, curvatures, contact stress

1. INTRODUCTION

Gear couplings are used to connect end of shafts and to compensate the misalignments. Most important components of them are the sleeve and the hub. The sleeve is an internal spur gear and the hub is an external gear which has crowned teeth. The two toothed components compose a special gear pair, wherein both number of teeth are the same. The gear coupling is able to compensate the misalignment of the coupled shafts by the tooth crowning and backlash. Using a single hub and sleeve, the effect of angular misalignment may be eliminated. In the practice, generally two hub-sleeve pairs are built up as it is shown in Figure 1. In this case the compensation of the offset misalignment is possible in addition to the angular misalignment.

A typical failure mode of the gear couplings is the surface fatigue, so called pitting formation, which can be prevented by limiting the contact stress. Due to the crowned surface of the hub, the surfaces of gear coupling having angular misalignment contact to each other in a point at any moment. To determine the contact stress need to know the location of the current contact point on the surface and the principal curvatures in that point. In this paper the mathematical models of tooth surfaces are prepared and the position of the contact points is determined for the gear coupling having angular misalignment operation. The principal curvatures and the principal directions of curvature are determined according to the involute tooth geometry for the surface of the sleeve, and by the method developed with the assistance of the envelope surface for the crowned surface of the hub. The load capacity of the coupling is based on the Hertz theory and it is determined by comparing the calculated and permissible contact stress.

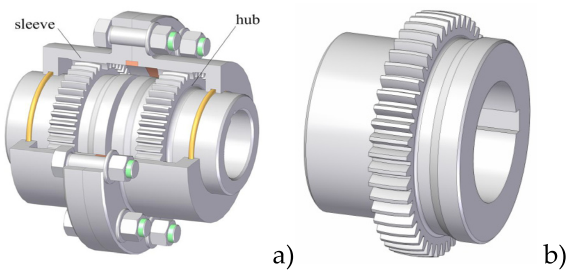

Figure 1. Gear coupling (a) & the hub with crowned teeth (b)

2. MATHEMATICAL MODELS FOR TOOTH SURFACES

2.1. Tooth surface of the hub

The crowned surface of hub is generally prepared by hobbing. Mathematical modeling of the production is made more solutions [3, 4, 5], which describe the surface with sufficient precision and the relation to one another shows minimal differences.

In this study, the tool surface is a straight circular cone (Figure 2), which has translational movement tangent to the pitch circle of hub with constant velocity $v_0 = r_1\omega_1$. Here r_1 is the radius of pitch circle on tooth surface of hub and ω_1 is the angular velocity of the hub.

The position vector of tool surface at point P is the following in the coordinate system S_0 (O_0, x_0, y_0, z_0):

$$\boldsymbol{r}_0 = \begin{bmatrix} u\sin\alpha \\ (R-u\cos\alpha)\cos\psi \\ (R-u\cos\alpha)\sin\psi \end{bmatrix}.$$

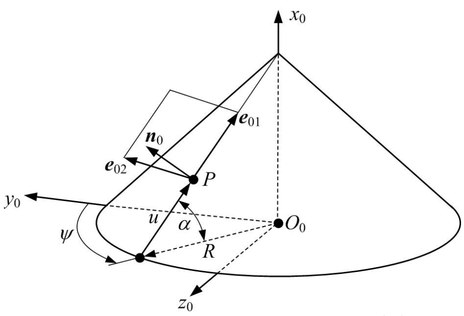

Figure 2. Conical tool surface

In equation (1) u and ψ are the parameters of tool surface, α is the profile angle and R is the crowning parameter. The interpretation of the notations is shown in Figure 2. Unit vectors for surface normal and two principal directions are indicated in the figure. They are described in coordinate system of tool surface by the following equations:

$$\boldsymbol{n}_0 = \begin{bmatrix}\cos\alpha & \sin\alpha\cos\psi & \sin\alpha\sin\psi\end{bmatrix}^T, \tag{2}$$

$$\boldsymbol{e}_{01} = \begin{bmatrix}\sin\alpha & -\cos\alpha\cos\psi & -\cos\alpha\sin\psi\end{bmatrix}^T, \tag{3}$$

$$\boldsymbol{e}_{02} = \begin{bmatrix}0 & \sin\psi & -\cos\psi\end{bmatrix}^T. \tag{4}$$

Here and later the superscript T means the transpose for vectors.

To determine the crowned tooth surface of the hub, the tool surface have to be described in coordinate system S_1 (O_1, x_1, y_1, z_1) which is rigidly connected to the hub. The transformation is based on Figure 3.

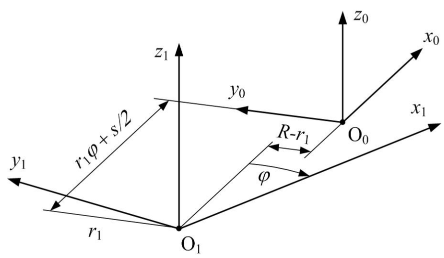

Figure 3. Coordinate systems

The moving tool surface provides a set of surface in system S_1, which is characterized by the following equation:

$$\boldsymbol{r}_1 = \boldsymbol{M}_{10}\begin{bmatrix} u\sin\alpha + r_1\varphi + s/2 \\ (R-u\cos\alpha)\cos\psi - R + r_1 \\ (R-u\cos\alpha)\sin\psi \end{bmatrix}, \tag{5}$$

where $\boldsymbol{M}_{10}$ is the transition matrix from system S_0 to system S_1, s is the tooth thickness of the crowned teeth, measured on the pitch circle in the central plane. The matrix is defined by the turning angle of the hub φ.

$$\boldsymbol{M}_{10} = \begin{bmatrix}\cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1\end{bmatrix}. \tag{6}$$

The equation of meshing is given by the following form:

$$\boldsymbol{v}_{01}\cdot\boldsymbol{n}_0 = 0. \tag{7}$$

Here $\boldsymbol{v}_{01}$ is the relative velocity between the hub and the tool surface, $\boldsymbol{n}_0$ is the unit normal vector in the contact point. Solving vector equation (7) a relationship is obtained between surface parameters (u and ψ) and the motion parameter φ, according to equation (8):

$$\varphi = -\frac{1}{r_1}\left(\frac{u}{\sin\alpha} + R\frac{1-\cos\psi}{\tan\alpha\cos\psi} + \frac{s}{2}\right). \tag{8}$$

Equations (5) and (8) together provide the crowned tooth surface. Details of derivation are found in reference [5]. The unit normal of the crowned tooth surface can be defined in coordinate system S_1 as follows:

$$\boldsymbol{n}_1 = \boldsymbol{M}_{10}\boldsymbol{n}_0 \tag{9}$$

2.2. Tooth surface of the sleeve

The sleeve is an internal spur gear which has involute tooth profile (see Figure 4). The position vector and unit normal vector of tooth surface are determined by the following equations:

$$\boldsymbol{r}_2 = \begin{bmatrix} r_b\left[-\sin(\vartheta-\eta)+\vartheta\cos(\vartheta-\eta)\right] \\ r_b\left[\cos(\vartheta-\eta)+\vartheta\sin(\vartheta-\eta)\right] \\ t \end{bmatrix}, \tag{10}$$

$$\boldsymbol{n}_2 = \begin{bmatrix} \cos(\vartheta-\eta) & \sin(\vartheta-\eta) & 0 \end{bmatrix}^T, \tag{11}$$

where r_b is the base circle radius, ϑ and t are the parameters of tooth surface, and η is the angle of tooth space measured on the base circle.

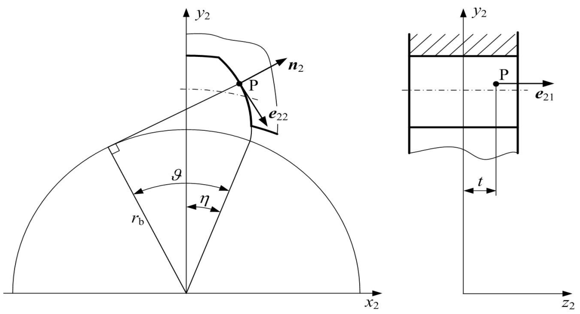
Figure 4. Internal tooth surface of the sleeve

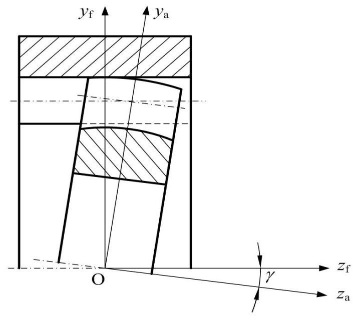
Figure 5. Gear coupling with angular misalignment

2.3. Determination of the contact points

The gear couplings having angular misalignment compose a special gear pair with intersecting axes (Figure 5). The shaft angle of the drive equals the angle of misalignment γ. The number of teeth on the hub equal to the number of teeth on the sleeve. The crowned tooth surface of the hub and the involute cylindrical tooth surface of the sleeve are in point contact in any instant.

The contact points are determined and the meshing is analyzed in a suitably chosen coordinate system. The contact points are the common points of both tooth surfaces therefore the position vectors are the same in these points. In addition the two tooth surfaces have common tangent plane in the contact points thus the normal vectors coincide to each other.

2.4. Coordinate systems

We set up four coordinate systems. S_1 (O, x_1, y_1, z_1) and S_2 (O, x_2, y_2, z_2) are moving coordinate systems and they are rigidly connected to the hub (gear 1) and the sleeve (gear 2), respectively. S_f (O, x_f, y_f, z_f) and S_a (O, x_a, y_a, z_a) are stationary coordinate systems fixed to the frame. S_f is the global system and S_a is an auxiliary ones. If there is no angular misalignment (γ = 0) S_a coincides with S_f (Figure 5). All of the coordinate systems have a common origin O. S_1 rotates in S_a around axis z_a which coincides with z_1. The turning angle ϕ_1 is measured between axes x_a and x_1 (Figure 6). When ϕ_1 = 0 S_1 coincides with S_a. Similarly, S_2 rotates in S_f around axis z_f which coincides with z_2. The turning angle ϕ_2 is measured between axes x_f and x_2 (see Figure 6). When ϕ_2 = 0 S_2 coincides with S_f. The relationships of transformation among coordinate systems and the transition matrices are described in detail in reference [6].

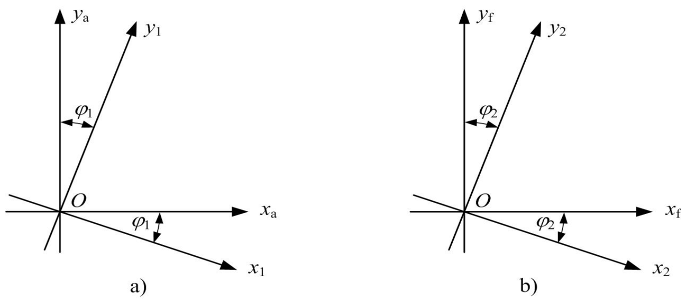
Figure 6. Applied coordinate systems

2.5. Contact points of the tooth surfaces

The contact point between the tooth surfaces of the hub and the sleeve is a point in the coordinate system S_f, at which the position vectors and the surface unit normals coincide with another. Thus

$$\boldsymbol{r}_f^{(1)}(u,\psi,\varphi_1) = \boldsymbol{r}_f^{(2)}(\vartheta,t,\varphi_2)\,, \tag{12}$$

$$\boldsymbol{n}_f^{(1)}(u,\psi,\varphi_1) = \boldsymbol{n}_f^{(2)}(\vartheta,\varphi_2)\,. \tag{13}$$

Vector equation (12) yields three scalar equations, but equation (13) yields only two independent scalar equations because of the both surface normals are unit vector.

$$\left|\mathbf{n}_f^{(1)}\right| = \left|\mathbf{n}_f^{(2)}\right| = 1\,. \tag{14}$$

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The tooth surfaces of the hub and the sleeve are represented in coordinate systems S_1 and S_2 by the equations (5) and (10), respectively. To obtain equations (12) and (13) the position vectors and normal vectors have to be transformed into S_f fixed coordinate system. The details can be found in reference [6].

Equation system given by equations (12) and (13) contains five nonlinear equations. Solving is possible by computer in iterative way using numerical solution method.

2.6. Determination of principal curvatures

Tooth surfaces of the gears are commonly produced by generating, using the principle of surface enveloping. These surfaces are usually sufficiently complicated to determine the curvatures in a conventional manner, by the methods of differential geometry. *Litvin* [1] proposed an effective solution to solve the problem. The method is that, the curvature characteristics of the enveloped tooth surface are determined by the given curvature characteristics of the tool surface (principal curvatures, principal directions) and the motion parameters.

2.7. Principal curvatures of the hub

The principal curvatures and principal directions for crowned tooth surfaces of the hub are determined in accordance with the recommendation of reference [1] using the valid correlations for enveloping surfaces. The first step is to be produced the principal curvatures and the principal directions on the tool surface. The tooth surface of the hub is generated by a conical tool surface (Figure 2). The principal directions are plotted in Figure 2. One of the principal directions is located along the cone generatrix, the other is in the tangent plane perpendicular to the first one. The unit vectors of principal directions are described by the equations (3) and (4). Principal curvatures for the indicated directions are given by the following formulas:

$$k_{01} = 0, \tag{15}$$

$$k_{02} = -\frac{\sin\alpha}{R + u\cos\alpha}. \tag{16}$$

In equation (16) the negative sign indicates that the center of curvature lies on the opposite direction of the surface normal.

Knowing the principal curvatures and principal directions of the tool surface and using the motion parameters, the following equations can be produced according to [1]:

$$\sigma = \frac{1}{2}\arctan\frac{-2c_1c_2}{c_2^2 - c_1^2 - \left(k_{01} - k_{02}\right)c_3}, \tag{17}$$

$$k_{12} = \frac{1}{2}\left(k_{01} + k_{02} + \frac{c_1^2 + c_2^2}{c_3} + \frac{c_2^2 - c_1^2 - (k_{0I} - k_{0II})c_3}{c_3\cos 2\sigma}\right), \tag{18}$$

$$k_{11} = k_{01} + k_{02} + \frac{c_1^2 + c_2^2}{c_3} - k_{12} \tag{19}$$

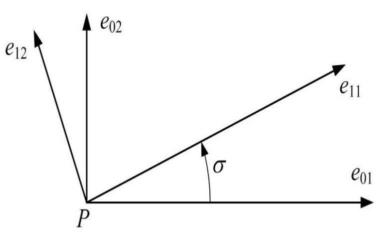

Figure 7. Principal directions of curvatures

Equation (17) determines the angle σ between the first principal direction of the tool surface $\boldsymbol{e}_{01}$ and the first principal direction of the workpiece tooth surface $\boldsymbol{e}_{11}$, which is interpreted according to Figure 7. Equations (18) and (19) give the sought principal curvatures of tooth surface. The following auxiliary variables are used:

$$c_1 = -k_{01}v_1 + \left(\boldsymbol{n}_0 \times \boldsymbol{\omega}_{01}\right)\boldsymbol{e}_{01}, \tag{20}$$

$$c_2 = -k_{02}v_2 + \left(\boldsymbol{n}_0 \times \boldsymbol{\omega}_{01}\right)\boldsymbol{e}_{02}, \tag{21}$$

$$c_3 = -k_{01}\left(v_1\right)^2 - k_{02}\left(v_2\right)^2 + \left(\boldsymbol{n}_0 \times \boldsymbol{\omega}_{01}\right)\boldsymbol{v}_{01} + \boldsymbol{n}_0\left(\boldsymbol{\omega}_1 \times \boldsymbol{v}_0\right). \tag{22}$$

In these expressions v_1 and v_2 are the components of relative velocity divided to the principal directions of the tool surface. $\boldsymbol{\omega}_{01}$ is the relative angular velocity vector. The following terms are used to define them:

$$v_1 = \boldsymbol{v}_{01}\boldsymbol{e}_{01}, \tag{23}$$

$$v_2 = \boldsymbol{v}_{01}\boldsymbol{e}_{02}, \tag{24}$$

$$\boldsymbol{\omega}_{01} = -\boldsymbol{\omega}_1 = \begin{bmatrix} 0 & 0 & \omega \end{bmatrix}^T. \tag{25}$$

Unit vectors of the principal direction for the tooth surface from Figure 7 are the following:

$$\boldsymbol{e}_{11} = \boldsymbol{e}_{01}\cos\sigma + \boldsymbol{e}_{02}\sin\sigma, \tag{26}$$

$$\boldsymbol{e}_{12} = -\boldsymbol{e}_{01}\sin\sigma + \boldsymbol{e}_{02}\cos\sigma. \tag{27}$$

Considering that these vectors are defined in the coordinate system S_0 they must be transferred to the own coordinate system of the hub S_1. After transformation, the principal directions of the tool surface are amended, but the relative position of them does not change relative to the principal direction of the tooth surface. Accordingly, the unit vectors of principal directions in the system S_1 as follows:

$$\boldsymbol{e}_{01}^{(1)} = \boldsymbol{M}_{10}\boldsymbol{e}_{01}, \tag{28}$$

$$\boldsymbol{e}_{02}^{(1)} = \boldsymbol{M}_{10}\boldsymbol{e}_{02}, \tag{29}$$

$$\boldsymbol{e}_{11}^{(1)} = \boldsymbol{e}_{01}^{(1)}\cos\sigma + \boldsymbol{e}_{02}^{(1)}\sin\sigma, \tag{30}$$

$$\boldsymbol{e}_{12}^{(1)} = -\boldsymbol{e}_{01}^{(1)}\sin\sigma + \boldsymbol{e}_{02}^{(1)}\cos\sigma. \tag{31}$$

The matrix of transformation $\boldsymbol{M}_{10}$ is given by equation (6).

2.8. Principal curvatures of the sleeve

Principal curvatures of involute cylindrical surface of the sleeve are known from involute geometry. One of the principal directions coincides with the generatrix and the corresponding curvature is

$$k_{21} = 0. \tag{32}$$

The other principal direction is tangent to the involute profile and the curvature in this plane equals the profile curvature. Based on Figure 4, we can write:

$$k_{22} = -1/r_b\vartheta. \tag{33}$$

The unit vectors of principal directions are:

$$\boldsymbol{e}_{21} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \tag{34}$$

$$\boldsymbol{e}_{22} = \begin{bmatrix} \sin(\vartheta - \eta) & -\cos(\vartheta - \eta) & 0 \end{bmatrix}^T. \tag{35}$$

2.9. Computing of the contact stress

The verification of the load capacity for tooth surfaces is carried out based on the Hertz theory. The tooth surfaces contact to each other in a point theoretically, but under load they are pressed, and the contact is formed along an elliptical pattern. The contact stress is determined by the following equation in accordance with the recommendation [2]:

$$\sigma_H = \frac{3}{2}\frac{F_n K_A}{\pi a b}, \tag{36}$$

where F_n is the normal force, K_A is the application factor, a and b are the two semi-axes of the contact ellipse.

The application factor depends on the type of the driving and driven machine. Its value should be selected based on the coupling manufacturer's recommendations.

Determining the semi-axes of the contact ellipse, the following relationships are used:

$$a = a^*\sqrt[3]{\frac{3}{2}\frac{F_n}{E_r\Sigma k}}, \text{ and } b = b^*\sqrt[3]{\frac{3}{2}\frac{F_n}{E_r\Sigma k}} \tag{37}$$

where a^* and b^* are the specific semi-axes of contact ellipse, E_r is the reduced modulus of elasticity, Σk is the sum of principal curvature.

The specific semi-axes can be determined from the diagram of Figure 8 as a function of curvature relation. In the Figure 8, θ is an auxiliary parameter depends on ratio of curvature $F(k)$. The value of θ must be defined in degrees:

$$\theta = \arccos\left(F(k)\right). \tag{38}$$

The ratio of curvature depends on the principal curvatures and the angle σ_{12} between the first principal directions of the two bodies:

$$F(k)=\frac{\sqrt{(k_{11}-k_{12})^2+2(k_{11}-k_{12})(k_{21}-k_{22})\cos 2\sigma_{12}+(k_{21}-k_{22})^2}}{\Sigma k}. \tag{39}$$

To calculate the parameters E_r and Σk in equations (37) and (39) the following formulas are used:

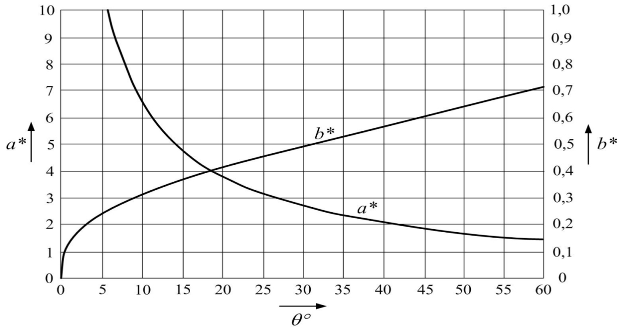

Figure 8. Specific semi-axes of contact ellipse

$$\frac{1}{E_r}=\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}, \tag{40}$$

$$\Sigma k=k_{11}+k_{12}+k_{21}+k_{22}. \tag{41}$$

In equation (40) E_1 and E_2 are the modulus of elasticity and ν_1 and ν_2 are the Poisson's ratio of contact bodies. The principal curvatures using in equation (41) was determined in section 4.

The permissible contact stress according to [2] is:

$$\sigma_{HP}=\frac{\sigma_{HN}Z_G}{s_H}, \tag{42}$$

where σ_{HN} is the rated contact strength, Z_G is the sliding coefficient, and s_H is the safety factor. σ_{HN} depends on the load cycles and it is equal to the endurance limit or the life strength. Z_G depends on the sliding velocity and we can compute it using the recommendation of reference [2]. The safety factor should be chosen depending on the application, the minimum suggested value of $s_{H\min}=1$.

The load carrying capacity of coupling is determined by the comparison of the computed and the allowed contact stress:

$$\sigma_H \le \sigma_{HP}. \tag{43}$$

Assuming equality in formula (43) a transcendental equation is obtained in F_n, which can be solved using numerical methods. The torque can be transmitted by the coupling given by the following equation:

$$T=Z_\varepsilon F_n r_b, \tag{44}$$

where Z_ε is the contact ratio factor, r_b is the base circle radius of the hub tooth surface. Theoretical value of Z_ε equals 2.

3. CONCLUSION

To prevent the surface fatigue on tooth surface of gear couplings the contact stress has to be limited. To determine the contact stress, we need to know the equations of tooth surfaces, the relationships of meshing, and the curvatures at the contact points. The tooth surfaces were prepared in accordance with the methods of manufacture. In the analysis of the operation, to find the points of contact we have had a non-linear system of equations having five equations. In the calculation of the load carrying capacity of gear coupling also became necessary the numerical solution for nonlinear equations. Solving the nonlinear equations, the possibilities of Mathcad program were used.

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