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LAPLACE DECOMPOSITION METHOD (LDM) FOR SOLVING NONLINEAR GAS DYNAMIC EQUATION

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Abstract: Laplace Decomposition Method (LDM), an efficient numerical method has been successfully tested for analytic solution of various nonlinear partial differential equations (PDE). In this paper, we discuss the application of LDM to find analytic and approximate solution of a nonlinear gas dynamic equation. Examples with different initial profiles are considered to compare the results and illustrate the effectiveness of the method. In this paper, LDM is used to find analytic and approximate solutions of nonlinear gas dynamic equation which arises in modeling of diverse physical phenomena.
Keywords: NonlinearPartial Differential Equations, Analytic Solution, Laplace Decomposition Method

1. INTRODUCTION

Nonlinearity plays a vital role in the understanding the most of physical, chemical, biological, and engineering sciences[1].A variety of analytical and numerical models have been developed over the last few decades to provide a framework for understanding the nonlinear gas dynamic wave phenomena.Thesemethods haveundergoneseveralmodifications for better results and accuracy. Laplace Decomposition Method(LDM), which uses Laplace transform and Adomian decomposition method ADM (George Adomian, 1986) to solve nonlinear gas dynamic equations is very well suited for physical problems where linearization, perturbation etc are not involved. LDM was first proposed by Suheil A.Khuri (2001) [1] and subsequently developed by many researchers. This method has been successfully applied to solve diverse nonlinear problems [2-8]. In this paper, LDM is used to find analytic and approximate solutions of nonlinear gas dynamic equation

$$u_t + uu_x + u = 0\,,\ x \in \mathbb{R} \text{ and } t>0 \tag{1}$$

which arises in modeling of diverse physical phenomena.

2. LAPLACE DECOMPOSITION METHOD(LDM)

We consider the following general form of nonlinear non-homogeneous partial differential equation (PDE) for illustrating the basic idea of the LDM.

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \quad with\ initial\ condition\ u(x,0) = h(x) \tag{2}$$

where $D=\frac{\partial}{\partial t}$ is the linear differential operator, R is the remaining linear operator of less order than D, N is non-linear differential operator and $g(x,t)$ is source term.

Operating Laplace transform on (2), we get

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \tag{3}$$

Using differential property of Laplace transform on (3) and by using initial condition, we get

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{1}{s}L[g(x,t)] - \frac{1}{s}L[Ru(x,t)] - \frac{1}{s}L[Nu(x,t)] \tag{4}$$

In LDM it is assumed that the solution $u(x,t)$ and the nonlinear term $Nu(x,t)$ can be expressed by the infinite series of the following forms

$$u(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t) \tag{5}$$

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \tag{6}$$

A_n, the Adomian's polynomials, are determined by the following relation

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$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \text{ n=0,1,2,........} \quad (7)$$

Substituting the expressions of (5) and (6) in (4) and then applying inverse Laplace transform we can calculate Adomian polynomials $A_n(u)$and the components of $u_n(x,t)$ and hence the solution u(x,t).

For more information on LDM, readers can refer[1, 9].

3.APPLYING LDM TO GAS DYNAMIC EQUATION

Example 1. Consider nonlinear gas dynamic equation with initial condition as follows.

$$u_t + uu_x + u = 0, u(x,0) = x \quad (8)$$

Applying Laplace transform, we get

$$su(x,s) - u(x,0) + L(uu_x) + u(x,s) = 0, \quad (9)$$

By using initial condition $u(x,0) = x$, we can write

$$u(x,s) = \frac{x}{s+1} - \frac{1}{s+1} L\left(\sum_{n=0}^{\infty} A_n(u)\right), \text{where } \sum_{n=0}^{\infty} A_n(u) = uu_x \quad (10)$$

Apply inverse Laplace transform to get the solution

$$u(x,t) = xe^{-t} - L^{-1}\left(\frac{1}{s+1} L\left(\sum_{n=0}^{\infty} A_n(u)\right)\right), \quad (11)$$

whereAdomian polynomials $A_n(u)$ can be calculated by using(7). It can be clearly seen that,

$$u_0(x,t) = xe^{-t} \quad (12)$$

First few polynomials and few terms of decomposition for the solution $u(x,t)$ are obtained by using mathematical software MATLAB as below:

$$A_0(u) = u_0 u_{0x}$$
$$u_1(x,t) = x(e^{-2t} - e^{-t})$$
$$A_1(u) = u_0 u_{1x} + u_{0x} u_1$$
$$u_2(x,t) = x(e^{-t} - 2e^{-2t} + e^{-3t})$$
$$A_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_{0x} u_2$$
$$u_3(x,t) = x(3e^{-2t} - e^{-t} - 3e^{-3t} + e^{-4t})$$
$$A_3(u) = u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_{0x} u_3$$
$$u_4(x,t) = x(e^{-t} - 4e^{-2t} + 6e^{-3t} - 4e^{-4t} + e^{-5t})$$
$$A_4(u) = u_0 u_{4x} + u_1 u_{3x} + u_2 u_{2x} + u_3 u_{1x} + u_{0x} u_4,$$

Substituting these terms in (5) i.e. $u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$,

we get

$$u(x,t) = x(e^{-t} + e^{-2t} - e^{-t} + e^{-t} - 2e^{-2t} + e^{-3t} + 3e^{-2t} - e^{-t} - 3e^{-3t} + e^{-4t} + \cdots) \quad (13)$$

By rearrangement and simplification of (13), we get,

$$u_0(x,t) = xe^{-t} = xe^{-t}(1 - e^{-t})^0$$
$$u_1(x,t) = x(e^{-2t} - e^{-t}) = -xe^{-t}(1 - e^{-t})^1$$
$$u_2(x,t) = x(e^{-t} - 2e^{-2t} + e^{-3t}) = xe^{-t}(1 - e^{-t})^2$$
$$u_3(x,t) = x(3e^{-2t} - e^{-t} - 3e^{-3t} + e^{-4t}) = -xe^{-t}(1 - e^{-t})^3$$
$$\text{So, } u(x,t) = xe^{-t} - xe^{-t}(1 - e^{-t})^1 + xe^{-t}(1 - e^{-t})^2 - xe^{-t}(1 - e^{-t})^3 + \cdots$$
$$u(x,t) = xe^{-t}(1 - (1 - e^{-t})^1 + (1 - e^{-t})^2 - (1 - e^{-t})^3 + \cdots)$$
$$u(x,t) = xe^{-t}(1 + 1 - e^{-t})^{-1}$$

The exact solution of (8) as

$$u(x,t) = \frac{xe^{-t}}{(2 - e^{-t})} \quad (14)$$

Remark: It is to be noted that the rearrangement of (13) in different way, we get

$$u(x,t) = x(e^{-t} + e^{-2t} - e^{-t} + e^{-t} - 2e^{-2t} + e^{-3t} + 3e^{-2t} - e^{-t} - 3e^{-3t} + e^{-4t} + \cdots)$$
$$\text{i.e. } u(x,t) = xe^{-t}(1 + e^{-t} + e^{-2t} + e^{-3t} + \cdots)$$
$$u(x,t) = xe^{-t}(1 - e^{-t})^{-1}$$

The exact solution,

$$u(x,t) = \frac{xe^{-t}}{(1 - e^{-t})} \quad (15)$$

With the help of MATLAB the graphs of initial wave profile (Figure 1) and exact solutions (Figure 2 and Figure 3) are drawn. The graphs clearly indicate that LDM is an efficient method to find the analytic solution of nonlinear partial differential equation, provided the initial condition is given.

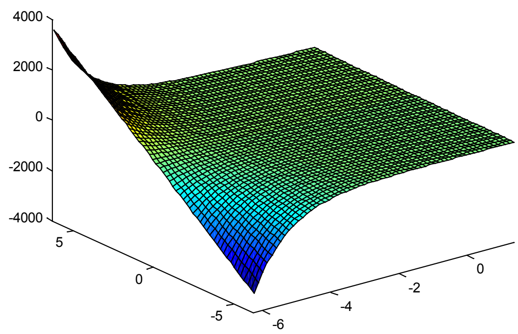

Figure 1. Surface plot of initial profile $u_0(x,t) = xe^{-t}$

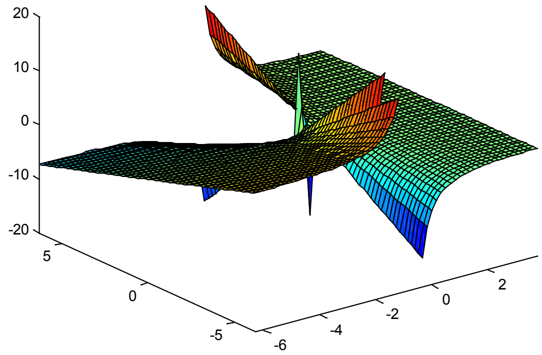

Figure 2. Surface plot of exact solution of $u(x,t) = \frac{xe^{-t}}{(2-e^{-t})}$

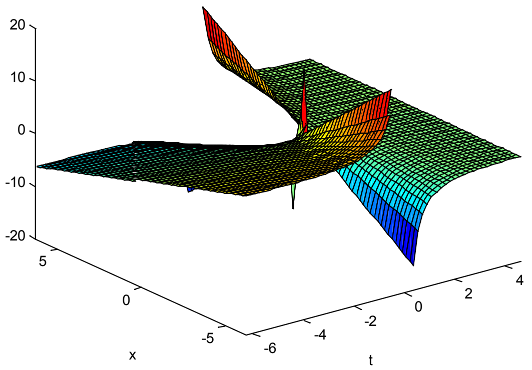

Figure 3. Surface plot of exact solution of $u(x,t) = \frac{xe^{-t}}{(1-e^{-t})}$

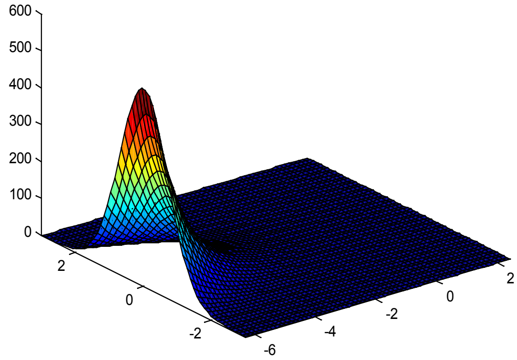

Figure 4. Surface plot of initial profile $u_0(x,t) = e^{-t-x^2}$

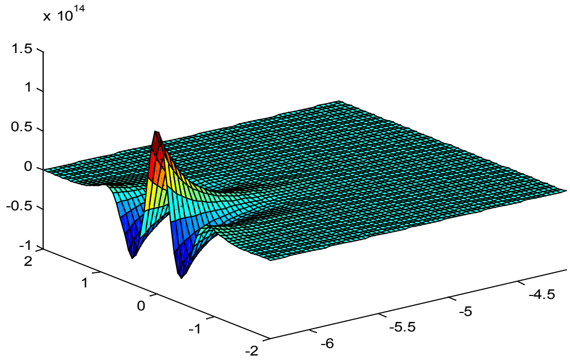

Figure 5. Surface plot of approximate solution $u(x,t) = \sum_{n=0}^{4} u_n\,(x,t)$

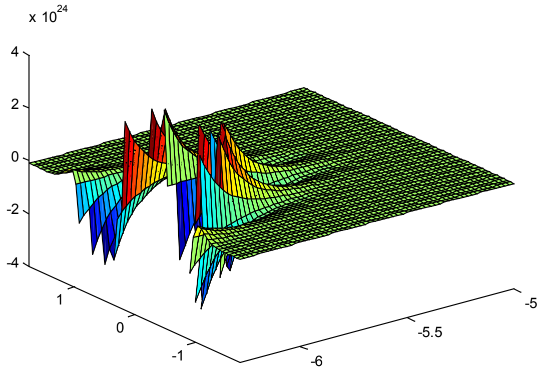

Figure 6. Surface plot of approximate solution $\mathrm{u(x,t)} = \sum_{\mathrm{n=0}}^{9} \mathrm{u_n\,(x,t)}$

Example 2. Consider the same nonlinear gas dynamic equation with different initial condition as follows.

$$u_t + uu_x + u = 0, u(x,0) = e^{-x^2} \tag{16}$$

Proceeding as in the example 1, we get decomposition for the solution as below.

$$u_0(x,t) = e^{-t-x^2}$$

$$u_2(x,t) = e^{-2t}\left(2e^{-3x^2} - 12x^2e^{-3x^2}\right) - e^{-3t}\left(e^{-3x^2} - 6x^2e^{-3x^2}\right) - e^{-t}\left(e^{-3x^2} - 6x^2e^{-3x^2}\right)$$

By substituting first few terms in (5), we get

$$u(x,t) = e^{-t-x^2} + 2xe^{-t-2x^2}(1-e^{-t})^1 + (6x^2-1)e^{-t-3x^2}(1-e^{-t})^2 + \cdots$$

We have calculated first 10 terms by using MATLAB software. So, approximate series solution for $u(x,t)$ will be

$$u(x,t) \cong u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$$

$$u(x,t) \cong e^{-t-x^2} + e^{-t}\left(36xe^{-6x^2} - 288x^3e^{-6x^2}\right) + \cdots - 2xe^{-2t-2x^2}$$

The graphs of the initial wave profile (Figure 4) and approximate solution by taking decomposition of first five terms (Figure 5) and then ten terms (Figure 6) have been drawn. The graphs help us to understand the wave propagation and behavior of (16).

5. CONCLUSION

In this paper, LDM is successfully applied for solving nonlinear gas dynamic equation,$u_t + uu_x + u = 0$, with different initial conditions, which arises in the study of various physical phenomena. Adomian's polynomials and series solution of the examples considered in this paper are computed by using MATLAB. Graphs are drawn to understand the wave profile of the equation. These results show that,LDM is effective and powerfulmethod for solving nonlinear PDE.

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