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[1.] Dávid GÖNCZI

THERMAL STRESSES IN THIN ANISOTROPIC HOLLOW CIRCULAR DISK

[1.] Institute of Applied Mechanics, University of Miskolc, Miskolc, HUNGARY

ABSTRACT: The thin anisotropic elastic hollow circular disk is subjected to radial steady-state temperature field. It will be assumed that there is identical convective heat exchange on the lower and upper plane boundary surfaces and the conditions of the generalized plane-stress is satisfied. The time independent temperatures and pressures are prescribed on the inner and outer curved boundary surfaces. This paper presents an analytical solution in which the solution of the heat conduction equation is approximated by a rational function.
Keywords: anisotropic, circular disk, thermal stresses, steady-state

1. INTRODUCTION

The determination of thermal stresses and displacements caused by axisymmetric steady-state temperature field in thin isotropic homogeneous and inhomogeneous elastic disk is the object of several works.

Some textbooks such as Timoshenko and Goodier [1], Solecki and Conant [2], Barber [3], Baroumi and Ragab [4], Hetnarski and Eslami [5], Noda et. al [6] give detailed analysis of the thermal stress problem for homogeneous isotropic elastic disk with axisymmetric temperature field. In papers by Pen, X. and Li, X. [7] the thermoelastic problem of isotropic functionally graded disk with arbitrary radial nonhomogeneity is considered. The numerical solution of the steady-state thermoelastic problem is reduced to a solution of a Fredholm integral equation [7]. A general analysis of one dimensional steady-state thermal stresses in thick cylinder made of isotropic radially inhomogeneous elastic material is analysed by Jabbari et. al [8]. An analytical method is used to solve the heat conduction and Navier equations in [8].

In the present paper we consider thin anisotropic homogeneous hollow circular disk which is subjected to axisymmetric steady-state temperature field. It is assumed that there is identical convective heat exchange on the lower and upper plane surfaces of the anisotropic disk. Books and papers [1-8] neglect the convective heat exchange on the surfaces $z=\pm t$ (Figure 1).

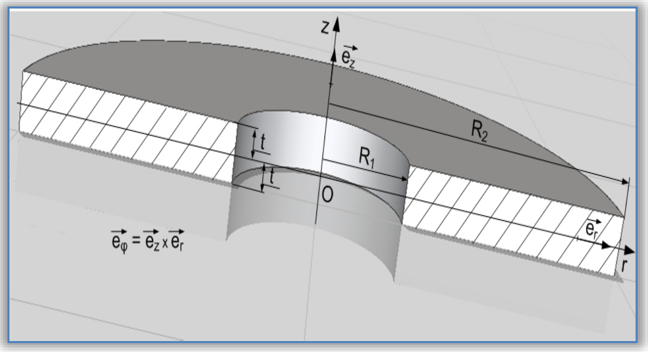

Figure 1. The sketch of the thin anisotropic disk

2. FORMULATION OF THE AXISYMMETRIC STEADY-STATE THERMAL STRESS PROBLEM AND ITS SOLUTION

The governing equations of the considered steady-state thermal stress problem are formulated in cylindrical coordinate system ($O_{r\varphi z}$). The material of the elastic circular disk is cylindrical anisotropic and it is assumed that the conditions of generalized plane stress are satisfied [9]. The stress-strain relation for cylindrical anisotropic elastic material including thermal effect for generalized plane stress state can be written in the form [9-10]:

$$\varepsilon_r = S_{11}\sigma_r + S_{12}\sigma_\varphi + S_{16}\tau_{r\varphi} + \alpha_1 T\,, \tag{1}$$

$$\varepsilon_\varphi = S_{12}\sigma_r + S_{22}\sigma_\varphi + S_{26}\tau_{r\varphi} + \alpha_2 T\,, \tag{2}$$

$$\gamma_{r\varphi} = S_{16}\sigma_r + S_{26}\sigma_\varphi + S_{66}\tau_{r\varphi} + \alpha_6 T\,. \tag{3}$$

In equations (1-3) ε_r, ε_φ are normal strains, $\gamma_{r\varphi}$ denotes the shear strain, σ_r and σ_φ are normal stresses, $\tau_{r\varphi}$ is shearing stress, $T=\tau-\tau_0$ is the temperature difference, τ is the absolute temperature, τ_0 is the reference temperature, S_{ij} $(i,j=1,2,6)$ is elastic compliance coefficient and α_i $(i=1,2,6)$ is the coefficient of linear thermal expansion. We look for the axisymmetric solution of the steady-state thermal stress problem which means that all quantities depend only on the radial coordinate. Internal heat source and body forces are not present and the following boundary conditions are prescribed (Figure1).

$$\sigma_r(R_1) = -p_1,\ \ \sigma_r(R_2) = -p_2, \tag{4}$$

$$\tau_{r\varphi}(R_1) = \tau_{r\varphi}(R_2) = 0, \tag{5}$$

$$T(R_1) = T_1,\ \ T(R_2) = T_2. \tag{6}$$

For the generalized plane stress state when the stresses depend on only the radial coordinate the equations of mechanical equilibrium are as follows

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \tag{7}$$

$$\frac{d\tau_{r\varphi}}{dr} + \frac{2\tau_{r\varphi}}{r} = 0. \tag{8}$$

From equation (8) it follows that

$$\tau_{r\varphi} = \frac{F}{r^2}. \tag{9}$$

Combination of Eq. (4) with Eq. (9) gives

$$F = 0, \quad \tau_{r\varphi} = 0. \tag{10}$$

Introducing Eq. (10) into Eqs. (1), (2) and expressing σ_r and σ_φ in terms of ε_r, ε_φ and T we obtain

$$\sigma_r = C_{11}\varepsilon_r + C_{12}\varepsilon_\varphi + \beta_1 T, \tag{11}$$

$$\sigma_\varphi = C_{12}\varepsilon_r + C_{22}\varepsilon_\varphi + \beta_2 T, \tag{12}$$

where

$$C_{11} = \frac{S_{22}}{S},\ C_{12} = -\frac{S_{12}}{S},\ C_{22} = \frac{S_{11}}{S}, \tag{13}$$

$$\beta_1 = \frac{S_{12}\alpha_2 - S_{22}\alpha_1}{S}, \quad \beta_2 = \frac{S_{12}\alpha_1 - S_{11}\alpha_2}{S}, \tag{14}$$

$$S = S_{11}S_{22} - S_{12}^2. \tag{15}$$

For the present problem the strain - displacement relationship can be formulated as [1, 10]

$$\varepsilon_r = \frac{du}{dr},\ \ \varepsilon_\varphi = \frac{u}{r},\ \ \gamma_{r\varphi} = \frac{dv}{dr} - \frac{v}{r}, \tag{16}$$

where $u=u(r)$ is the radial displacement and $v=v(r)$ is the tangential displacement. Substitution of Equations (11), (12) and $(16)_{1,2}$ into Eq. (7) gives the following differential equation for $u=u(r)$:

$$C_{11}\frac{d^2u}{dr^2} + C_{11}\frac{1}{r}\frac{du}{dr} - C_{22}\frac{u}{r^2} + \beta_1\frac{dT}{dr} + (\beta_1 - \beta_2)\frac{T}{r} = 0\,. \tag{17}$$

The true temperature function which is the solution of the corresponding heat conduction equation is approximated (to avoid numerical problems and reduce the time of the numerical solution) as

$$T(r) \approx \Theta(r) = \vartheta_2 r^2 + \vartheta_1 r^1 + \vartheta_0 + \vartheta_{-1} r^{-1} + \vartheta_{-2} r^{-2}. \tag{18}$$

The next section of this paper deals with the determination of $T=T(r)$ and its approximation $\vartheta=\vartheta(r)$. We replace Eq. (17) by the following equation:

$$C_{11}\frac{d^2u}{dr^2} + C_{11}\frac{1}{r}\frac{du}{dr} - C_{22}\frac{u}{r^2} + \beta_1\frac{d\Theta}{dr} + (\beta_1 - \beta_2)\frac{\Theta}{r} = 0\,. \tag{19}$$

After lengthy derivation [11] the general solution of Eq. (19) can be written in the next form:

$$u(r)=K_1r^c+K_2r^{-c}+\vartheta_2\frac{\beta_2-3\beta_1}{9C_{11}-C_{22}}r^3+\vartheta_1\frac{\beta_2-2\beta_1}{4C_{11}-C_{22}}r^2+\vartheta_0\frac{\beta_2-\beta_1}{C_{11}-C_{22}}r- \\ -\vartheta_{-1}\frac{\beta_2}{C_{22}}+\vartheta_{-2}\frac{\beta_2+\beta_1}{C_{11}-C_{22}}r^{-1}, \tag{20}$$

where

$$c^2=\frac{C_{22}}{C_{11}}. \tag{21}$$

In Eq. (20) K_1 and K_2 are the constants of the integration. The combination of Eq. (20) yields the formula of normal stresses

$$\sigma_r(r)=K_1(cC_{11}+C_{12})r^{c-1}+K_2(-cC_{11}+C_{12})r^{-c-1}+\vartheta_2\frac{\beta_2-3\beta_1}{9C_{11}-C_{22}}(3C_{11}+C_{12})r^2+ \\ +\vartheta_1\frac{\beta_2-2\beta_1}{4C_{11}-C_{22}}(2C_{11}+C_{12})r+\vartheta_0\frac{\beta_2-\beta_1}{C_{11}-C_{22}}(C_{11}+C_{12})-\vartheta_{-1}\frac{\beta_2}{C_{22}}C_{12}r^{-1}+ \\ +\vartheta_{-2}\frac{\beta_2+\beta_1}{C_{11}-C_{22}}(-C_{11}+C_{12})r^{-2}+\beta_1(\vartheta_2r^2+\vartheta_1r^1+\vartheta_0+\vartheta_{-1}r^{-1}+\vartheta_{-2}r^{-2}), \tag{22}$$

$$\sigma_\varphi(r)=K_1(cC_{12}+C_{22})r^{c-1}+K_2(-cC_{12}+C_{22})r^{-c-1}+\vartheta_2\frac{\beta_2-3\beta_1}{9C_{11}-C_{22}}(3C_{12}+C_{22})r^2+ \\ +\vartheta_1\frac{\beta_2-2\beta_1}{4C_{11}-C_{22}}(2C_{12}+C_{22})r+\vartheta_0\frac{\beta_2-\beta_1}{C_{11}-C_{22}}(C_{12}+C_{22})-\vartheta_{-1}\beta_2r^{-1}+ \\ +\vartheta_{-2}\frac{\beta_2+\beta_1}{C_{11}-C_{22}}(-C_{12}+C_{22})r^{-2}+\beta_2(\vartheta_2r^2+\vartheta_1r^1+\vartheta_0+\vartheta_{-1}r^{-1}+\vartheta_{-2}r^{-2}). \tag{23}$$

By the use of Eqs. (3), (10) and Eq. $(16)_3$ we can write

$$\frac{dv}{dr}-\frac{v}{r}=r\frac{d}{dr}\left(\frac{v}{r}\right)=S_{16}\sigma_r+S_{26}\sigma_\varphi+\alpha_6T. \tag{24}$$

From Eq. (24) we obtain for the tangential displacement $v=v(r)$ the following expression:

$$v(r)-v(R_1)=r\int_{R_1}^{r}S_{16}\frac{\sigma_r(r)}{r}+S_{26}\frac{\sigma_\varphi(r)}{r}+\alpha_6\frac{T(r)}{r}\,dr. \tag{25}$$

In Eq. (25) $v(R_1)$ describes a rigid body rotation about axis z which may be an arbitrary value.

3. DETERMINATION OF THE TEMPERATURE FIELD

The hollow circular anisotropic disk is shown in Figure 1. Its thickness is $2t$ and it is assumed that there is identical convective heat exchange on the plane boundary surfaces $z=\pm t$. Axisymmetric steady-state temperature field is caused by given surface temperature on the cylindrical boundary $r=R_1$ and $r=R_2$ ($|z|\le t$). The surrounding medium has zero temperature and the prescribed boundary conditions in our problem are

$$T(R_1)=T_1,\quad T(R_2)=T_2=0. \tag{26}$$

According to the Fourier's law of heat conduction we have

$$q_r=-\lambda_{11}\frac{\partial T}{\partial r}-\lambda_{12}\frac{1}{r}\frac{\partial T}{\partial\varphi}, \tag{27}$$

$$q_\varphi=-\lambda_{12}\frac{\partial T}{\partial r}-\lambda_{22}\frac{1}{r}\frac{\partial T}{\partial\varphi}, \tag{28}$$

where λ_{11}, λ_{12} and λ_{22} are the coefficients of thermal conductance of cylindrical anisotropic material. We consider only axisymmetric temperature field. This means that

$$\frac{\partial T}{\partial\varphi}=0. \tag{29}$$

It was mentioned that there is no internal heat source and the material of the disk is homogeneous. In the present problem the heat flux vector

$$\mathbf{q}=\mathbf{q}(r,\varphi)=q_r(r,\varphi)\mathbf{e_r}+q_\varphi(r,\varphi)\mathbf{e_\varphi} \tag{30}$$

satisfies the next equation [12]:

$$\nabla\cdot(t\mathbf{q})+hT=t\left(\frac{\partial q_r}{\partial r}+\frac{q_r}{r}+\frac{1}{r}\frac{\partial q_\varphi}{\partial\varphi}\right)+hT=0. \tag{31}$$

In Eqs. (30), (31) e_r adn e_φ are the unit vectors in radial and circumferential directions and

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\mathbf{e}_\varphi, \tag{32}$$

h is the heat transfer coefficient by convection and the dot between the vectors in Eq. (31) denotes their scalar product. The combination of Eqs. (27), (28), (29) with Eq. (31) leads to the result

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - p^2 T = 0, \tag{33}$$

where

$$p^2 = \frac{h}{t\lambda}. \tag{34}$$

The general solution of the (33) differential equation is as follows [5, 11]

$$T(r) = k_1 I_0(pr) + k_2 K_0(pr), \tag{35}$$

where k_1 and k_2 are the constants of integration. In Eq. (35) $I_0(x)$ is the modified Bessel function of the first kind and order zero and $K_0(x)$ is the corresponding function of the second kind. For the numerical computations the $T=T(r)$ function is replaced by $\vartheta=\vartheta(r)$ which is obtained from $T=T(r)$ by the method of least squares approximation [13].

4. COMPUTATION OF THE CONSTANTS OF INTEGRATION

We limited to our analysis to the case of stress boundary conditions

$$\sigma_r(R_1) = -p_1, \;\; \sigma_r(R_2) = -p_2 = 0. \tag{36}$$

Let $\sigma(r)$ be defined as

$$\sigma(r) = \vartheta_2 \frac{\beta_2 - 3\beta_1}{9C_{11} - C_{22}}(3C_{11} + C_{12})r^2 + \vartheta_1 \frac{\beta_2 - 2\beta_1}{4C_{11} - C_{22}}(2C_{11} + C_{12})r + \vartheta_0 \frac{\beta_2 - \beta_1}{C_{11} - C_{22}}(C_{11} + C_{12}) -$$
$$-\vartheta_{-1}\frac{\beta_2}{C_{22}}C_{12}r^{-1} + \vartheta_{-2}\frac{\beta_2 + \beta_1}{C_{11} - C_{22}}(-C_{11} + C_{12})r^{-2} + \beta_1(\vartheta_2 r^2 + \vartheta_1 r^1 + \vartheta_0 + \vartheta_{-1}r^{-1} + \vartheta_{-2}r^{-2}), \tag{37}$$

Using the formula of $\sigma_r(r)$ and Eqs. (36), (37) we get for the constants K_1 and K_2

$$K_1 = \frac{f_1 a_{22} - f_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \qquad K_2 = \frac{f_2 a_{11} - f_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{38}$$

where

$$a_{11} = (cC_{11} + C_{12})R_1^{c-1}, \qquad a_{12} = (-cC_{11} + C_{12})R_1^{-c-1}, \tag{39}$$

$$a_{21} = (cC_{11} + C_{12})R_2^{c-1}, \qquad a_{22} = (-cC_{11} + C_{12})R_2^{-c-1}, \tag{40}$$

$$f_1 = -p - \sigma(R_1), \qquad f_2 = -\sigma(R_2). \tag{41}$$

From Eqs. (26), (35) it follows that

$$k_1 = \frac{K_0(pR_2)}{I_0(pR_1)K_0(pR_2) - I_0(pR_2)K_0(pR_1)}T_1, \tag{42}$$

$$k_2 = -\frac{I_0(pR_2)}{I_0(pR_1)K_0(pR_2) - I_0(pR_2)K_0(pR_1)}T_1. \tag{43}$$

5. NUMERICAL EXAMPLE

The following data are used for the numerical computation:

$$R_1 = 0.02\ m, \quad R_2 = 0.08\ m, \quad t = 0.0015\ m, \; \lambda_{11} = 50\frac{W}{mK}, \quad h = 70\frac{W}{m^2K}, \quad T_1 = 150°C,$$

$$\alpha_1 = 18.73\cdot 10^{-6}\frac{1}{K}, \alpha_2 = 11.981\cdot 10^{-6}\frac{1}{K}, \alpha_6 = -11.69\cdot 10^{-6}\frac{1}{K}, S_{11} = 0.8053\cdot 10^{-10}\frac{m^2}{N},$$

$$S_{12} = -0.7878\cdot 10^{-11}\frac{m^2}{N}, \quad S_{16} = -0.3243\cdot 10^{-10}\frac{m^2}{N}, \quad S_{22} = 0.3475\cdot 10^{-10}\frac{m^2}{N},$$

$$S_{26} = -0.4696\cdot 10^{-10}\frac{m^2}{N}, \quad S_{66} = 0.1141\cdot 10^{-10}\frac{m^2}{N}.$$

The temperature function $T=T(r)$ denoted with blue solid line, $\vartheta=\vartheta(r)$ with red dash line are shown in Figure 2/a and the errors of the applied approximation are indicated in Figure 2/b. The definition of error functions $e_1=e_1(r)$ and $e_2=e_2(r)$ are as follows

$$e_1(r) = \left|\frac{\Theta(r) - T(r)}{T(r)}\right|, \qquad e_2(r) = \left|\frac{\frac{d\Theta(r)}{dr} - \frac{dT(r)}{dr}}{\frac{dT(r)}{dr}}\right|. \tag{44}$$

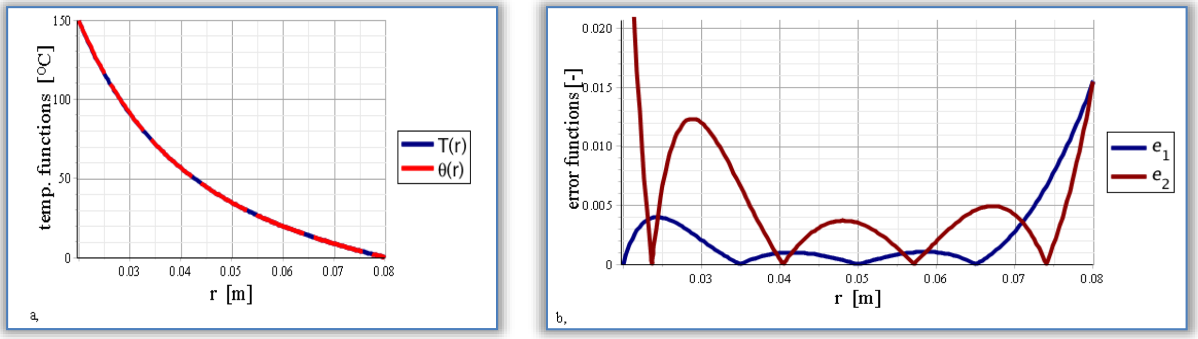

Figure 2. The temperature function T(r), its approximation θ(r) and the error functions (e_1(r) and e_2(r)).

According to the computations the error of the approximated temperature function is under 1.5% (Figure 2).We considered three numerical examples. First we investigated the original problem (θ(r)≠0K, T_1=150K, p_1=20MPa and denoted by red solid lines in the diagrams: Figure 3-Figure 6), then the heat load free case (blue dash line, θ(r)=0K, p_1=20MPa) and in the last case we dealt with the steady-state thermal stress problem when p_1=0MPa, θ(r)≠0K and T_1=150K (green dashdot line). Figure 3 indicates the solutions for the displacemet fields of this cases.

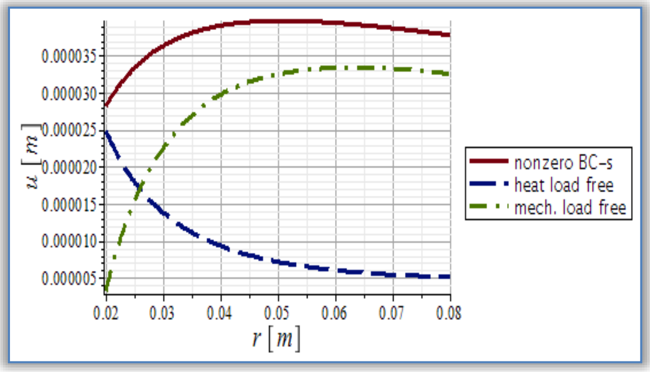

Figure 3. The solutions of the displacement fields

The calculated radial stresses can be seen in Figure 4 and the σ_φ normal stresses are shown in Figure 5 as the function of the radial coordinate. Figure 6 illustrates the solutions for the tangential displacements of the investigated cases.

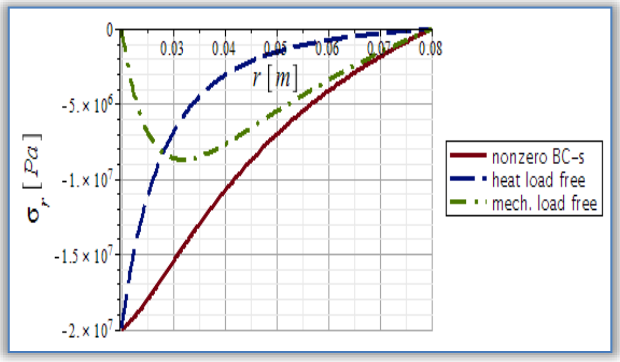

Figure 4. The solutions of the radial stresses of the investigated cases

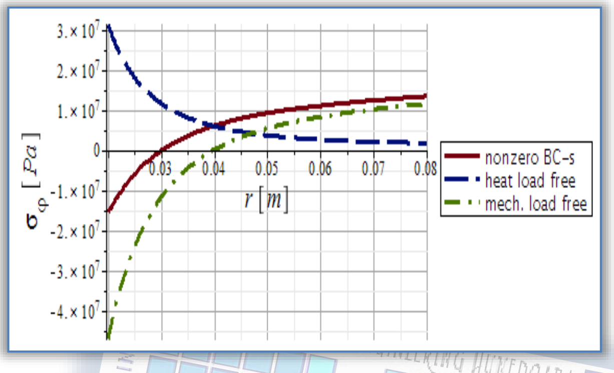

Figure 5. The plots of the tangential normal stresses

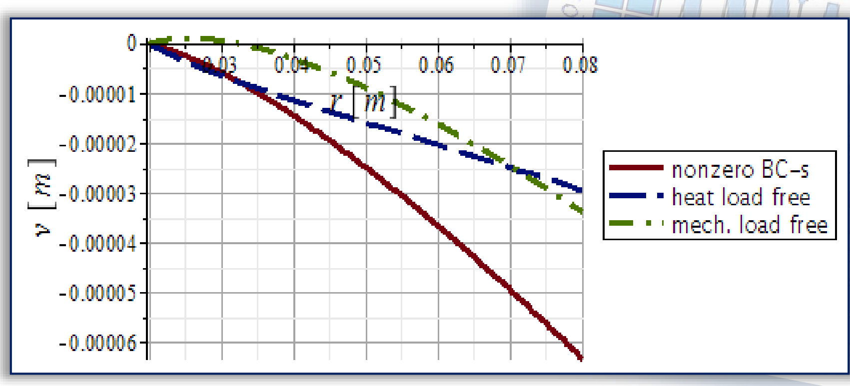

Figure 6. The solutions for the tangential displacement ($v(r)$)

6. CONCLUSIONS

This paper presents an approximate analytical solution for the determination of a steady-state thermal stress problem. The thin elastic hollow circular disk is subjected to radial steady-state temperature field. The material of the considered disk is cylindrical anisotropic. It is assumed that there is identical convective heat exchange on the lower and upper plane boundary surfaces and the conditions of the generalized plane stress state is satisfied. The presented analytical solution is based on the least squares approximation of the temperature field. Examples illustrate the application of the developed approximate analytical methods.

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