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QUASI-STATIC TRANSIENT THERMAL STRESSES IN A THICK ANNULAR PLATE SUBJECTED TO SECTIONAL HEAT SUPPLY

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ABSTRACT: The principal aim of this paper is to investigate the thermoelastic problems in a thick annular plate subjected to sectional heat supply on the upper surfaces whereas the fixed circular edges are at zero temperature. The governing heat conduction equation has been solved by using integral transform technique. The results are obtained in series form in terms of Bessel's functions. The results for displacement and stresses have been computed numerically and illustrated graphically.
Keywords: thick annular plate, thermoelasticity, integral transform

1. INTRODUCTION

As a result of the increased usage of industrial and construction materials the interest in the thermal stress problems has grown considerably, typified by the annular fins of heat exchangers and brake disc rotors, because of its elementary geometry. Therefore, a number of theoretical studies concerning them have been reported so far. For example, Nowacki [6] has determined steady-state thermal stresses in circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and the circular edge. The direct thermoelastic problem in an annular fin is studied by Wu [10] investigates the transient thermal stresses in an annular fin with its base subjected to a heat flux of a decayed exponential function of time. Wankhede [11] has determined the quasi-static thermal stresses in thin circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Gogulwar and Deshmukh [3] solved the inverse problem of thermal stresses in a thin annular disc, which was further generalized [2] in direct problem. Chiu and Chen [1] investigated stress-field in an annular fin of temperature-dependent conductivity under a periodic heat transfer boundary condition is analyzed by the Adomian's decomposition method. Recently Ootao et al. [8] performed analysis of a three-dimensional transient thermal stress problem is developed for a nonhomogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and /or outer surfaces. In this paper, our attempt has been made to discuss quasi-static transient thermal stresses in a thick annular plate $a \le r \le b$, $-h \le z \le h$ and the result illustrated numerically and graphically by using integral transform technique. No one previously studied such type of problem. This is a new contribution to the field.

2. FORMULATION OF THE PROBLEM

Consider a thick annular plate of thickness 2h, occupying a space D defined by $a \le r \le b$, $-h \le z \le h$. Let the plate be subjected to a transient asymmetric temperature field on the axial direction & axisymmetric temperature field on the radial direction of the cylindrical coordinate system. Initially the plate is kept at zero temperature the arbitrary heat flux $Qf(r)/\lambda$ is prescribed over the upper surface (z = h) and the lower surface (z = -h) the fixed circular edge (r = a and r = b) are at zero temperature. Assume the upper and lower surface of thick annular plate are traction–free surface under this realistic prescribed condition the quasi-static transient thermal stresses are required to be determined.

2.1. Temperature distribution

The transient heat conduction equation is given as follows

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

in which thermal diffusivity of the material of the plate is denoted as $\kappa = \lambda / \rho C$, λ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant, subjected to the initial and boundary conditions as

$$T = 0 \text{ at } t = 0 \tag{2}$$

$$T = 0 \text{ at } r = a, \ -h \le z \le h \ , \ t > 0 \tag{3}$$

$$T = 0 \text{ at } r = b, \ -h \le z \le h \ , \ t > 0 \tag{4}$$

$$T = 0 \text{ at } z = -h, \ a \le r \le b \ , \ t > 0 \tag{5}$$

$$\frac{\partial T}{\partial z} = (Q/\lambda) f(r) t, \text{ at } z = h, \ a \le r \le b \ , \ t > 0 \tag{6}$$

2.2. Thermal displacements and thermal stress

The Navier's equations in the absence of body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [5]

$$\begin{aligned} &\nabla^2 u_r - \frac{u_r}{r} + \frac{1}{1-2\upsilon}\frac{\partial e}{\partial r} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial r} = 0 \\ &\nabla^2 u_z - \frac{1}{1-2\upsilon}\frac{\partial e}{\partial z} - \frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t \frac{\partial \theta}{\partial z} = 0 \end{aligned} \tag{7}$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential ϕ and Love's function L as [4]

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z}, u_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial^2 z} \tag{8}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t T \tag{9}$$

and the Love's function L must satisfy the equation

$$\nabla^2(\nabla^2 L) = 0 \tag{10}$$

in which $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$

The component of the stresses are represented by the use of the potential ϕ and Love's function L as

$$\begin{aligned} &\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\upsilon \nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\}, \sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\upsilon \nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right)\right\}, \\ &\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}, \sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \end{aligned} \tag{11}$$

in which G and υ are the shear modulus and Poisson's ratio respectively.

The boundary condition on the traction free surface stress functions are

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } z = \pm h \tag{12}$$

Equations (1) to (16) constitute the mathematical formulation of the problem.

3. SOLUTION OF THE PROBLEM

3.1. Solution for Temperature distribution

Applying Laplace transformation [9] of the equation (1) to (6) with respect to t and using the equation (2) one obtain

$$\frac{\partial^2 \bar{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{T}}{\partial r}+\frac{\partial^2 \bar{T}}{\partial z^2}=\frac{P}{K}\bar{T} \tag{13}$$

with boundary condition

$$\frac{\partial \bar{T}}{\partial z}=\frac{Qf(x)}{\lambda P^2} \quad \text{at} z\ = h \tag{14}$$

$$\bar{T}=0 \quad \text{at } z=-h \tag{15}$$

$$\bar{T}=0 \quad \text{at} r = a \text{ and} r = b \tag{16}$$

where p is Laplace transform parameter and $\bar{T}$ Laplace transform of T

Introducing theHankel transform over the variable r and its inverse transformation defined [7] as

$$\bar{\bar{T}}(\alpha_m, z, p)=\int_a^b r K_0(\alpha_m r)\bar{T}(r,z,p)\,dr,$$

$$\bar{T}(r,z,p)=\sum_{n=1}^{\infty}\bar{\bar{T}}(\alpha_m, z, p)K_0(\alpha_m r) \tag{17}$$

in which $K_0(\alpha_m r)=\frac{R_0(\alpha_m r)}{\sqrt{N}}, R_0(\alpha_m, r)=\frac{J_0(\alpha_m r)}{J_0(\alpha_m b)}-\frac{Y_0(\alpha_m r)}{Y_0(\alpha_m b)}, N=(b^2/2)R_0'^2(\alpha_m b)-(a^2/2)R_0'^2(\alpha_m a),$

and $\alpha_1, \alpha_2 ..$ are roots of the transcendental equation $R_0(\alpha_m, a)=0$ with $J_n(x)$ is the Bessel function of the first kind of order n and $Y_n(x)$ is the Bessel function of the second kind of order n.

Applying the finite Hankel integral transform, and its inversion theorems for both transforms, yield

$$T=\left(\frac{Q}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)}{2h\sqrt{N}}\right)\wp_n R_0(\alpha_m, r) \tag{18}$$

in which $q^2=\alpha_m^2+P/K,\ \hbar_m=\alpha_m^2+(2n+1)^2\pi^2/16h^2,$

$$\wp_n=\sin\left[(z+h)\frac{(2n+1)\pi}{4h}\right]\int_0^t \tau\exp\{-k\hbar_m(t-\tau)\}d\tau\Big/\sin\left[(2n+1)\frac{\pi}{2}\right]$$

and $\bar{f}(\alpha_m)$ is the Hankel transform of $f(r)$.

3.2. Solution for thermal stresses

(a) Goodier thermoelastic displacement potential ϕ.

Referring to the fundamental equation (1) and its solution (18) for the heat conduction problem, the solution for the displacement function are represented by the Goodier's thermoelastic displacement potential ϕ governed by equation (9) are represented by

$$\phi=-\left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)}{\sqrt{N}}\right)\wp_{n\,\hbar_m}R_0(\alpha_m, r) \tag{24}$$

(b) Love's function L

Similarly, the solution for Love's function L are assumed so as to satisfy the governed condition of equation (12) as

$$L=\left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{\bar{f}(\alpha_m)\wp\alpha_{mn}}{\sqrt{N}}\right)R_0(\alpha_m, r)[H_{mn}\cosh[\alpha_m(z+h)]+R_{mn}\alpha_m(z+h)\sinh[\alpha_m(z+h)] \tag{25}$$

in which H_{mn} and R_{mn} are arbitrary unknown functions

(c) Displacement and Thermal stresses

In this manner, two displacement functions in the cylindrical coordinate system ϕ and L are fully formulated. Now, in order to obtain the displacement components, we substitute the values of thermoelastic displacement potential ϕ and Love's function L in equations (9) and (10), one obtains

$$U_r = \left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0'(\alpha_m, r)\{\wp_n / \hbar_m - [\alpha_m^2 H_{mn} \sinh[\alpha_m (z+h)] + R_{mn}\alpha_m^3 (z+h)\cosh[\alpha_m (z+h)]\} \quad (26)$$

$$U_z = -\left(\frac{QK}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0(\alpha_m, r)\{(2n+1)\wp_n \pi \cot\left[(z+h)\frac{(2n+1)\pi}{4h}\right] / 4h\hbar_m$$
$$-(\alpha_m^2 H_{mn}\cosh[\alpha_m (z+h)]) - R_{mn}[(z+h)\alpha_m^3 \sinh[\alpha_m (z+h)] - 2(1-2\upsilon)] \quad (38)$$
$$\times (z+h)\alpha_m^2 \cosh[\alpha_m (z+h)]]\}$$

$$\sigma_{rr} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\,\hbar_m} R_0''(\alpha_m, r) - \frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\,\hbar_m} R_0(\alpha_m, r)\right.$$
$$+ H_{mn}\langle R_0'(\alpha_m, r)\alpha_m \cosh[\alpha_m (z+h)]\rangle + R_{mn}\langle R_0'(\alpha_m, r) 2\alpha_m^3 \sinh[\alpha_m (z+h)] \quad (39)$$
$$\left. + R_0''(\alpha_m, r)[\alpha_m^2 (z+h)\cosh[\alpha_m (z+h)] + \sinh[\alpha_m (z+h)]\rangle\right\}$$

$$\sigma_{\theta\theta} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0'(\alpha_m, r) - \frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}\hbar_m} R_0(\alpha_m, r)\right.$$
$$+ H_{mn}\alpha_m^2 \langle R_0'(\alpha_m, r)\sinh[\alpha_m (z+h)]\rangle / r + R_{mn}\alpha_m^2 \langle 2\upsilon\alpha_m R_0(\alpha_m, r)\sinh[\alpha_m (z+h)] \quad (40)$$
$$\left. + r^{-1} R_0'(\alpha_m, r)(\sinh[\alpha_m (z+h)] + \alpha_m (z+h)\cosh[\alpha_m (z+h)])\rangle\right\}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}} R_0(\alpha_m, r) - \left(\frac{QK}{2h\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)\wp_n}{\sqrt{N}} R_0(\alpha_m, r)\right. \quad (41)$$
$$\left. - \alpha_m^3 H_{mn}\sinh[\alpha_m (z+h)] + \alpha_m^3 R_{mn}\langle (1-2\upsilon)\sinh[\alpha_m (z+h)] - \alpha_m (z+h)\cosh[\alpha_m (z+h)]\rangle\right\}$$

$$\sigma_{rz} = 2G\left\{\frac{QK}{2h\lambda}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\bar{f}(\alpha_m)}{\sqrt{N}} R_0'(\alpha_m, r)\times\frac{(2n+1)\pi\alpha_m}{4h\hbar_m}\cot\left[(z+h)\frac{(2n+1)\pi}{4h}\right] + H_{mn}\alpha_m^3 \cosh[\alpha_m (z+h)]\right. \quad (42)$$
$$\left. + R_{mn}(2\upsilon\cosh[\alpha_m (z+h)] + \alpha_m (z+h)\sinh[\alpha_m (z+h)])\right\}$$

(d) Determination of unknown arbitrary function H_{mn} and R_{mn}

Applying boundary condition (17) to the equation (25) and (28) one obtains

$$H_{mn} = 0 \quad (43)$$

$$R_{mn} = \frac{(2n+1)\pi}{4h\alpha_m^2 2\upsilon\hbar_m}\frac{\wp_n}{\sin\left[(z+h)\frac{(2n+1)\pi}{4h}\right]} \quad (44)$$

4. SPECIAL CASE AND NUMERICAL CALCULATIONS

Setting

$$f(r) = (r^2 - a^2)(r^2 - b^2) \quad (45)$$

Applying finite Hankel transform as defined in equation (21) to the equation (45), one obtain

$$\bar{f}(\alpha_m) = \frac{8\{(a^2\alpha_m^2 - 3b^2\alpha_m^2 + 16)J_0(\alpha_m a) - (b^2\alpha_m^2 - 3a^2\alpha_m^2 + 16)J_0(\alpha_m b)\}}{\pi\sqrt{N}\alpha_m^6 J_0(\alpha_m a)J_0(\alpha_m b)Y_0(\alpha_m b)} \quad (46)$$

5. NUMERICAL CALCULATIONS

The numerical calculation have been carried out for (SN 50C) plate with the parameters a = 1m, b = 2m, h = 0.3m, thermal diffusivity $k = 15.9 * 10^{-6}(m^2 s^{-1})$ and Poisson ratio v = 0.281 with $\alpha_1 = 3.120, \alpha_2 = 6.2734, \alpha_3 = 9.4182, \alpha_4 = 12.5614, \alpha_5 = 15.7040$ being the Positive roots of transcendental equation $R_0(\alpha_m, a) = 0$ For convenience setting $A = QK / \pi\lambda 10^5$, $B = 2GQK / \pi\lambda 10^5$ in the expression (3.39) to (3.44) The numerical expression for temperature, displacement and stress components are obtained by equations (34) and (37) to (42).In order to examine the influence of heat flux on the upper and lower surface of thick plate, one performed the numerical calculations r = 1, 1.2, 1.4, 1.6, 1.8, 2 m and z =- 0.3, -0.15, 0, 0.15, 0.3 m. and t = 5, Numerical variations in radial and axial directions are shown in the figures.

6. CONCLUDING REMARKS

In this study, we have treated thermoelastic problem of a thick annular plate which is considered traction free. We successfully established and obtained the expressions for temperature distribution, displacement and stress function due asymmetric arbitrary heat flux. Then, in order to examine the validity of boundary value problem, we analyze, as a particular case with mathematical model for $f(r) = (r^2 - a^2)(r^2 - b^2)$ and numerical calculations were carried out. The thermoelastic behavior is examined such as temperature, displacement and stresses with the help of arbitrary heat flux at upper surface applied.

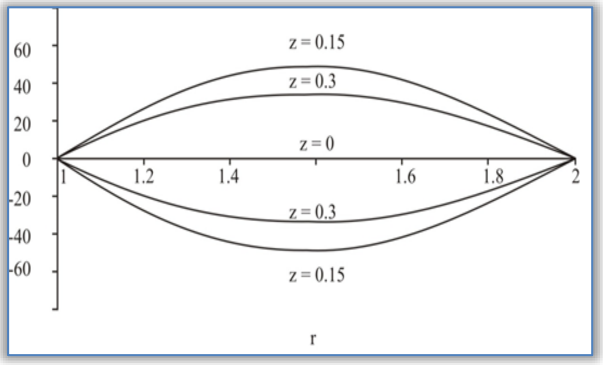

Figure 1: Axial displacement profile along axial direction

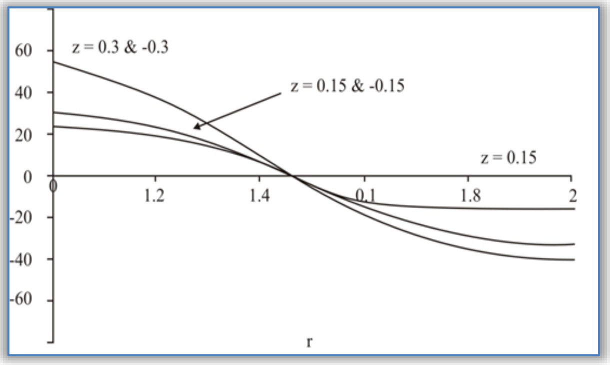

Figure 2: Radial displacement profile along axial direction

Figure 1 shows the axial displacement u_z occurs at the center i.e. r = 1.5 in radial direction where as in radial direction decreases from lower surface to upper surface.

As shown in Figure 2 the variation of thermal stress in the radial displacement u_r decreases from inner circular surface to outer circular surface in radial direction where as in axial direction it take place at upper and lower surfaces of the plate.

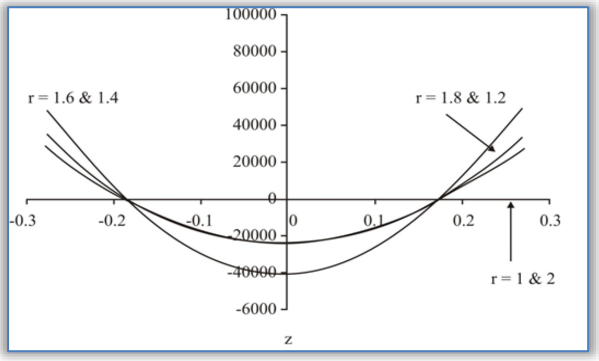

Figure 3: Radial stress distribution along radial direction

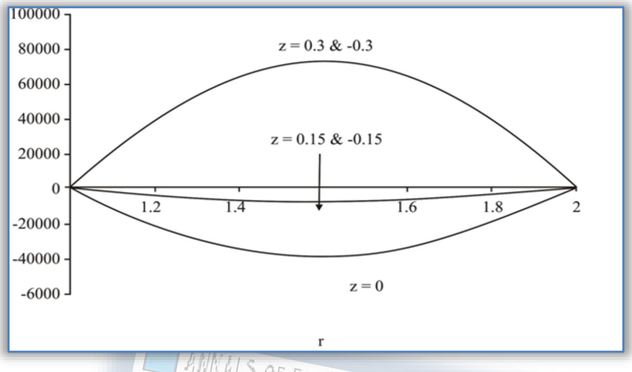

Figure 4: Radial stress distribution along axial direction

Figure 3 and 4 shows the radial stress function σ_{rr} develops tensile stress at upper and lower surface of the plate, where as it develop compressive stress in the middle of plate.

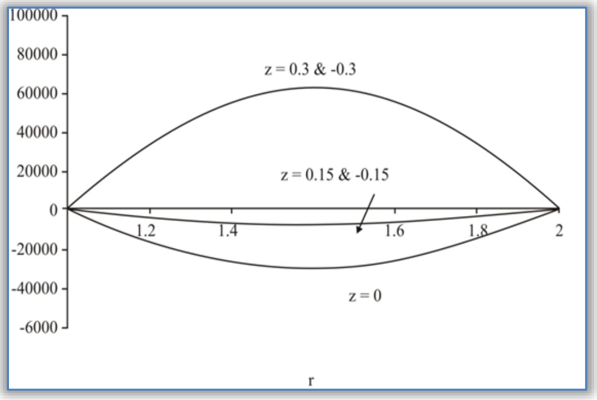

Figure 5: Tangential stress distribution along radial direction

Figure 5 shows the variation of the stress function $\sigma_{\theta\theta}$ develops tensile stress at the upper and lower surface of the plate where as it develops compressive stress in the middle of plate. We may conclude that the system of equations proposed in this study can be adapted to design of useful structures or machines in engineering applications in the determination of thermoelastic behavior at every instant and at all points of thick annular disc of finite height.

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