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APPROXIMATE SOLUTION OF ZAKHAROV-KUZNETSOV EQUATION VIA HOMOTOPY PERTURBATION METHOD

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Abstract: In this paper, Homotopy Perturbation Method (HPM) is applied to find the approximate solutions of ZK equations. It is proved that the HPM gives a powerful tool for solving a large number of nonlinear partial differential equations in mathematical physics. The solutions obtained by HPM are presented graphically.
Keywords: Zakharov-Kuznetsov (ZK) equation, homotopy perturbation method, approximate solutions

INTRODUCTION

The investigation of the travelling wave solutions for nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Most scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly. Except in a limited number of these problems are linear. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, Nano-Bioelectronics, plasma physics etc. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. One of the models is represented by the ZK equation. The Zakharov-Kuznetsov equation was introduced as an asymptotic model in [1] to describe the propagation of nonlinear ionic-sonic waves in magnetized lossless plasma in two dimensions. The physical phenomenon for this equation was investigated in [2-3]. A large number of evolution equations in many areas of applied mathematics, physics and engineering appear as a nonlinear wave equation. Most nonlinear equations are difficult to solve analytically, especially the ZK equation.

In recent years, many powerful methods are developed to find the exact solutions of the ZK equations such as tanh method, (G/G') method etc. [4-12]. In this present work, approximate solutions of two different types of ZK equation namely ZK (2, 2, 2) and ZK (3, 3, 3) are found by using the homotopy perturbation method. The homotopy perturbation method (HPM) was first proposed by He [13]. The HPM does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p\in[0,1]$ which is considered as a small parameter. Recently, many researchers do a lot of significant work about the application and the potential of homotopy perturbation method. The results are also shown graphically on mathematica.

HOMOTOPY PERTURBATION METHOD (HPM)

Consider the following non-linear differential equation

$$A(u)-f(r)=0 \quad r\in\Omega, \tag{1}$$

with the boundary conditions

$$B(u,\partial u/\partial n)=0 \quad r\in\Gamma. \tag{2}$$

while A, B, f(r) and Γ are differential operator, boundary operator, known analytic function and the boundary of the domain Ω, respectively.

The operator A(u) can be divided into a linear part L(u) and a non-linear part N(u).

Therefore Eq. (1) can be rewritten as:

$$L(u)+N(u)-f(r)=0. \tag{3}$$

In case the nonlinear Eq. (1) has no small parameter, we can construct the following homotopy

$$H(v, p) = L(v) - L(u_0) + p\,L(u_0) + p[N(v) - f(r)] = 0. \qquad (4)$$

where p is called homotopy parameter.
According to the homotopy perturbation method, the approximation solution of Eq. (4) can be expressed as a series of the power p, i.e.

$$u = \lim_{p\to 1} (u_0+u_1+u_2+u_3+\ldots). \qquad (5)$$

when Eq. (5) corresponds to Eq. (4), becomes the approximate solution of Eq. (1).

NUMERICAL APPLICATIONS

In this section, we apply HPM for solving two different types of equations namely ZK (2, 2, 2) and ZK (3, 3, 3) with specific initial conditions. The results obtained from HPM are very effective and reliable.

Example 1: Consider the following ZK (2, 2, 2) Equation

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0, \qquad (6)$$

with the initial condition

$$u(x,y,0) = \frac{4}{3}\lambda \sinh^2(x+y). \qquad (7)$$

In operator form Eq. (6) can be written as

$$L(u) = -(u^2)_x - \frac{1}{8}(u^2)_{xxx} - \frac{1}{8}(u^2)_{yyx}, \qquad (8)$$

Taking inverse operator, we have

$$u(x,y,t) = \frac{4}{3}\lambda \sinh^2(x+y) - L^{-1}\left((u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx}\right), \qquad (9)$$

Let the solution of the Eq. (6), according to HPM be

$$u(x,y,t) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots \qquad (10)$$

Putting Eq. (10) into Eq. (9)

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots = \frac{4}{3}\lambda \sinh^2(x+y) - pL^{-1}\begin{pmatrix} (u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)^2_x \\ + \frac{1}{8}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)^2_{xxx} \\ + \frac{1}{8}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)^2_{yyx} \end{pmatrix}.$$

Equating powers of *p*

$$p^0, \qquad u_0(x,y,t) = \frac{4}{3}\lambda \sinh^2(x+y),$$

$$p^1, \qquad u_1(x,y,t) = -L^{-1}\left((u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx}\right),$$

$$u_1(x,y,t) = -\frac{32}{9}\sinh(x+y)\cosh(x+y)\left[10\cosh^2(x+y) - 7\right]\lambda^2 t.$$

$$p^2, \qquad u_2(x,y,t) = -L^{-1}\left((2u_0u_1)_x + \frac{1}{8}(2u_0u_1)_{xxx} + \frac{1}{8}(2u_0u_1)_{yyx}\right)$$

$$u_2(x,y,t) = \frac{64}{27}\begin{bmatrix} 1200\cosh^6(x+y) - 2080\cosh^4(x+y) \\ + 968\cosh^2(x+y) - 79 \end{bmatrix}\lambda^3 t^2,$$

⋮

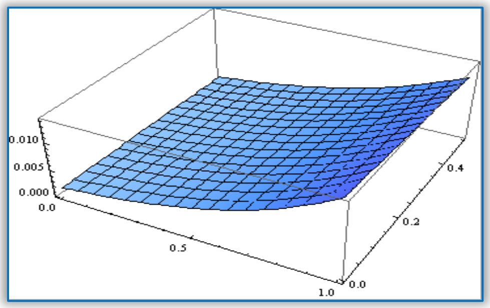

Figure 1. Graphical representation of approximate solution in the domain $t \in (0,0.5)$ and $x \in (0,1)$ when $y = 0.9$ and $\lambda = 0.001$.

and so on. The series solution of Eq. (6) is

$$u(x,y,t) = \frac{4}{3}\lambda \sinh^2(x+y) - \frac{32}{9}\sinh(x+y)\cosh(x+y)\left[10\cosh^2(x+y) - 7\right]\lambda^2 t + \ldots \qquad (11)$$

Example 2. Consider the following ZK (2, 2, 2) Equation

$$u_t + (u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx} = 0, \qquad (12)$$

with the initial condition

$$u(x,y,0) = \frac{4}{3}\lambda \cosh^2(x+y). \qquad (13)$$

In operator form Eq. (12) can be written as

$$L(u) = -(u^2)_x - \frac{1}{8}(u^2)_{xxx} - \frac{1}{8}(u^2)_{yyx}, \tag{14}$$

Taking inverse operator, we get

$$u(x,y,t) = -\frac{4}{3}\lambda \cosh^2(x+y) - L^{-1}\left((u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx}\right). \tag{15}$$

According to the above described procedure, we have

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots = -\frac{4}{3}\lambda\cosh^2(x+y) - pL^{-1}\begin{pmatrix}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_x^2 \\ +\frac{1}{8}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_{xxx}^2 \\ +\frac{1}{8}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_{yyx}^2\end{pmatrix}.$$

Consequently, we have

$$p^0, \quad u_0(x,y,t) = -\frac{4}{3}\lambda\cosh^2(x+y),$$

$$p^1, \quad u_1(x,y,t) = -L^{-1}\left((u^2)_x + \frac{1}{8}(u^2)_{xxx} + \frac{1}{8}(u^2)_{yyx}\right),$$

$$u_1(s,y,t) = -\frac{32}{9}\sinh(x+y)\cosh(x+y)\left[10\cosh^2(x+y) - 3\right]\lambda^2 t,$$

$$p^2, \quad u_2(x,y,t) = -L^{-1}\left((2u_0u_1)_x + \frac{1}{8}(2u_0u_1)_{xxx} + \frac{1}{8}(2u_0u_1)_{yyx}\right),$$

$$u_2(x,y,t) = -\frac{64}{27}\begin{bmatrix}1200\cosh^6(x+y) - 1520\cosh^4(x+y) \\ +408\cosh^2(x+y) - 9\end{bmatrix}\lambda^3 t^2.$$

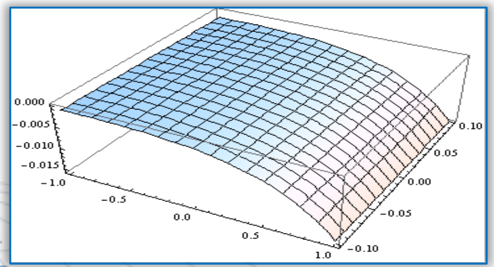

Figure 2. Graphical representation of approximate solution in the domain $t \in (-0.1, 0.1)$ and $x \in (-1,1)$ when $y = 0.9$, $\lambda = 0.001$.

and so on. The solution of Eq. (12) is

$$u(x,y,t) = -\frac{4}{3}\lambda\cosh^2(x+y) - \frac{32}{9}\sinh(x+y)\cosh(x+y)\left[10\cosh^2(x+y) - 3\right]\lambda^2 t + \ldots \tag{16}$$

Example 3. Consider the following ZK (3, 3, 3) Equation

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \tag{17}$$

with the initial condition

$$u(x,y,0) = \frac{3}{2}\lambda\sinh\left(\frac{1}{6}(x+y)\right). \tag{18}$$

Operator form of Eq. (17) can be written as

$$Lu = -(u^3)_x - 2(u^3)_{xxx} - 2(u^3)_{yyx}, \tag{19}$$

Applying inverse operator on Eq. (19), we have

$$u(x,y,t) = \frac{3}{2}\lambda\sinh(\frac{1}{6}(x+y)) - L^{-1}\left((u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx}\right) \tag{20}$$

According to HPM procedure, we have

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots =$$

$$\frac{3}{2}\lambda\sinh(\frac{1}{6}(x+y)) - pL^{-1}\begin{pmatrix}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_x^3 \\ +2(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_{xxx}^3 \\ +2(u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots)_{yyx}^3\end{pmatrix}.$$

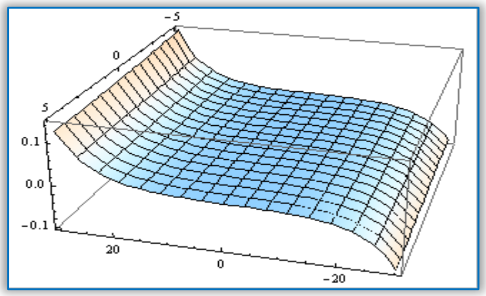

Figure 3. Graphical representation of approximate solution in the domain $t \in (-5,5)$ and $x \in (-30,30)$ when $y = 0.9$ and $\lambda = 0.001$.

Consequently, we have

$$p^0, \quad u_0(x,y,t) = \frac{3}{2}\lambda\sinh(\frac{1}{6}(x+y))$$

$$p^1, \quad u_1(x,y,t) = -L^{-1}\left[(u_0^3)_x + 2(u_0^3)_{xxx} + 2(u_0^3)_{yyx}\right]$$

$$u_1(x,y,t) = -\frac{3}{8}\cosh(\frac{1}{6}(x+y))\left[9\cosh^2(\frac{1}{6}(x+y)) - 8\right]\lambda^3 t$$

$$p^2, \quad u_2(x,y,t) = -L^{-1}\left[(3u_0^2u_1)_x + 2(3u_0^2u_1)_{xxx} + 2(3u_0^2u_1)_{yyx}\right]$$

$$u_2(x,y,t) = \frac{3}{64}\sinh(\frac{1}{6}(x+y))\begin{bmatrix}765\cosh^4(\frac{1}{6}(x+y)) - \\ 729\cosh^2(\frac{1}{6}(x+y)) + 91\end{bmatrix}\lambda^5 t^2$$

and so on. The solution of Eq. (17) is

$$u(x,y,t) = \frac{3}{2}\lambda\sinh(\frac{1}{6}(x+y)) - \frac{3}{8}\cosh(\frac{1}{6}(x+y))\left[9\cosh^2(\frac{1}{6}(x+y)) - 8\right]\lambda^3 t + \ldots \tag{21}$$

Example 4. Consider the following ZK (3, 3, 3) Equation

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \quad (22)$$

with the initial condition

$$u(x,y,0) = \frac{3}{2}\lambda \cosh(\frac{1}{6}(x+y)). \quad (23)$$

According to the above defined procedure, we have

$$u(x,y,t) = \frac{3}{2}\lambda \cosh(\frac{1}{6}(x+y)) - L^{-1}\left((u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx}\right). \quad (24)$$

Consequently, we have

$$p^0, \quad u_0(x,y,t) = \frac{3}{2}\lambda \cosh(\frac{1}{6}(x+y))$$

$$p^1, \quad u_1(x,y,t) = -L^{-1}\left[(u_0^3)_x + 2(u_0^3)_{xxx} + 2(u_0^3)_{yyx}\right]$$

$$u_1(x,y,t) = -\frac{3}{8}\sinh(\frac{1}{6}(x+y))\left[9\cosh^2(\frac{1}{6}(x+y)) - 1\right]\lambda^3 t$$

$$p^2, \quad u_2(x,y,t) = -L^{-1}\left[(3u_0^2u_1)_x + 2(3u_0^2u_1)_{xxx} + 2(3u_0^2u_1)_{yyx}\right]$$

$$u_2(x,y,t) = \frac{3}{64}\cosh(\frac{1}{6}(x+y))\begin{bmatrix} 765\cosh^4(\frac{1}{6}(x+y)) - \\ 801\cosh^2(\frac{1}{6}(x+y)) + 127 \end{bmatrix}\lambda^5 t^2$$

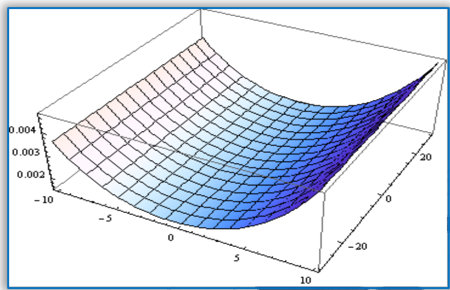

Figure 4. Graphical representation of approximate solution in the domain $t \in (-30,30)$ and $x \in (-10,10)$ when $y = 0.9$ and $\lambda = 0.001$.

and so on. The series solution of Eq. (22) is

$$u(x,y,t) = \frac{3}{2}\lambda \cosh(\frac{1}{6}(x+y)) - \frac{3}{8}\sinh(\frac{1}{6}(x+y))\left[9\cosh^2(\frac{1}{6}(x+y)) - 1\right]\lambda^3 t + \ldots \quad (25)$$

CONCLUSION

In this paper, we have used the Homotopy Perturbation Method (HPM) to derive the analytical approximate solutions of Zakharov-Kuznetsov (ZK) equation, especially for ZK (2, 2, 2) and ZK (3, 3, 3) equations with initial conditions. The results obtained from the proposed method are very accurate, efficient and reliable showing that HPM is very effective and powerful for solving the nonlinear partial differential equations. To reveal the convergence of the HPM, the results of the numerical example are presented and only few terms are required to obtain accurate solutions. These approximate solutions may provide a useful help for physicists to study more complex physical phenomena. Graphical representation depicts the compatibility of the proposed method with such complexity problems.

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