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EFFICIENT HOMOTOPY ANALYSIS METHOD TO SYSTEM OF DELAY DIFFERENTIAL EQUATIONS

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Abstract: This paper deal with the application of homotopy analysis method to solve the delay differential equations systems. This process provides speedy series convergence towards exact solution of the systems. Several Delayed problems of system are given in the direction of explaining the efficiency and validity of this technique. In this paper, we are going to effectually work with the homotopy analysis method to find the solution of delayed differential systems that contain nonlinearity. Solutions in the form of numeric lead the approximation towards exact form solution with higher accuracy. Homotopy analysis method will be presented in Section 2. Numerical systems are solved by Homotopy Analysis method and their graphical representation is presented in Section 3. The conclusion will be given n Section 4.
Keywords: Delay differential equations, Homotopy analysis method

1. INTRODUCTION

Delay differential equation models have wide and diverse range of applications. Hutchinson introduced first mathematical model of Delay in Biology for period maturation. Nonlinear delay differential equations used to model numerous chemical reactions, engineering problems, economical and biological systems. Delays show the possessions of transmission, transport processes, and inertia.As they consider inheritance of these. In 18th century, Laplace & Condorcet introduced delay differential equations [1]. After the Second World War development made in these equations, which still continues. Infinite spectrum of frequencies occurs in Delay problems. The detailed learning of collected works discloses that physical phenomena are nonlinear in nature and great need to get their solutions. Stepan& Insperger have used the semidiscretization method to conclude the permanency lobes of DDEs that model the dynamics of cutting machine operations.

First time in 1992, Liao developed Homotopy Analysis method and then many other people applied this method on the application of different systems [2-6].A variety of mathematical and physical problems have been solved by HAM [7]. Homotopy Analysis Method contain a parameter which is named as auxiliary parameter which provides quick convergence of the solution and due to this parameter it differ to other methods [8].This method doesn't depend upon any small or large parameters and is valid for most nonlinear models [9].The major advantage of HAM is also that in this method different base functions can be choose. Riccati equations, Vakhnenko equation [10], Glauert-jet equations [11], Hirota-Satsuma KdV equation [12], motion of projectile [13], boundary layer problems [14], Boltzmann equations [15], MKdV equation [16] and many more equations were efficiently solved by HAM.

2. HOMOTOPY ANALYSIS METHOD

For demonstration of the concept of HAM, by considering general non-linear problem

$$N[v(t)] = 0, \tag{1}$$

Non-linear operator is N and $v(t)$ is unidentified function with independent variable t.

2.1 Deformation equation of zeroth order

Eq (1) has initial guess of exact solution $v_0(t)$, auxiliary parameter $\hbar$, auxiliary function $H(t)$ & L is operator which is linear in its nature

$$L[g(t)] = 0, \qquad \text{when } g(t) = 0, \tag{2}$$

$$(1-q)L[\phi(t;q)-v_0(t)]-q\hbar H(t)N[\phi(t;q)]=0, \tag{3}$$

Embedding parameter q varies form 0 to 1, $\phi(t)$ is unknown function.

When $q=0$, deformation equation of zeroth order is

$$\phi(t;0)=v_0(0), \tag{4}$$

When $q=1$, deformation equation of zeroth order becomes

$$N[\phi(t;1)]=0, \tag{5}$$

This is exact solution. As variation in the parameter q, solution also goes closer to exact solution. Deformation derivatives of m[th] order is given as

$$\left.\frac{1}{m!}\frac{\partial^m\phi(t;q)}{\partial q^m}\right|_{q=0}, \tag{6}$$

And the solution in series form is

$$v(t)=v_0(t)+\sum_{m=1}^{\infty}v_m(t)q^m, \tag{7}$$

2.2 High – Order deformation equation

$$\vec{v}_m=v_0,v_1,v_2,\ldots,v_m, \tag{8}$$

After differentiating and diving by $m!$, deformation equation of m[th] order is

$$L[v_m(t)-\chi_m v_{m-1}(t)]=\hbar H(t)R_m[\vec{v}_{m-1}(t)], \tag{9}$$

and

$$\chi_m=\begin{cases}0 & \text{if } m\le 1\\ 1 & \text{if } m\ge 2\end{cases} \tag{10}$$

$$R_m[\vec{v}_{m-1}(t)]=\left.\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(t;q)}{\partial q^{m-1}}\right|_{q=0} \tag{11}$$

3. NUMERICAL EXAMPLES

This section contained solution of delayed systems of non-linear type solving by homotopy analysis method & comparison of results with graphical representation.

Example 3.1 Consider a system of delay differential equation of nonlinear type

$$\begin{cases}v_1'(t)=e^{-t}-e^{t/2}+v_1\left(\frac{t}{2}\right)+v_1(t)-v_2(t)\\ v_2'(t)=e^{t}+e^{t/2}-v_2\left(\frac{t}{2}\right)-v_1(t)-v_2(t),\end{cases} \tag{12}$$

with initial condition

$$\begin{cases}v_1(0)=1\\ v_2(0)=1,\end{cases} \tag{13}$$

We can choose freely initial approximation $v_0(t)$ to be

$$\begin{cases}v_{10}(t)=e^t\\ v_{20}(t)=e^t,\end{cases} \tag{14}$$

Deformation equation of zeroth order is

$$\begin{cases}(1-q)L[\phi_1(t;q)-v_{10}(t)]-q\hbar H(t)N[\phi_1(t;q)]=0\\ (1-q)L[\phi_2(t;q)-v_{20}(t)]-q\hbar H(t)N[\phi_2(t;q)]=0,\end{cases} \tag{15}$$

when $q=0$, Eq. (15) becomes

$$\begin{cases}L[\phi_1(t;q)-v_{10}(t)]=0\\ L[\phi_2(t;q)-v_{20}(t)]=0,\end{cases} \tag{16}$$

It gives initial approximation

$$\begin{cases}\phi_1(0;q)=v_{10}(0)=0\\ \phi_2(0;q)=v_{20}(0)=0,\end{cases} \tag{17}$$

When q=1 nonlinear terms are

$$\begin{cases} N[\phi_1(t;1)] = v_1'(t) - e^{-t} + e^{t/2} - v_1\left(\frac{t}{2}\right) - v_1(t) + v_2(t) \\ N[\phi_2(t;1)] = v_2'(t) - e^{t} - e^{t/2} + v_2\left(\frac{t}{2}\right) + v_1(t) + v_2(t), \end{cases} \quad (18)$$

Deformation equation of m^th order is

$$\begin{cases} L[v_{1m}(t) - \chi_m v_{1(m-1)}(t)] = \hbar H(t) R_{1m}[\vec{v}_{1(m-1)}(t)] \\ L[v_{21m}(t) - \chi_m v_{2(m-1)}(t)] = \hbar H(t) R_{2m}[\vec{v}_{2(m-1)}(t)], \end{cases} \quad (19)$$

By putting $\hbar = -1$, and $H(t) = 1$, in eq. (19), and

$$\begin{cases} R_{1m}[\vec{v}_{1(m-1)}(t)] = v'_{1(m-1)}(t) - v_{1(m-1)}(\frac{t}{2}) - v_{1(m-1)}(t) + v_{2(m-1)}(t) - (1-\chi_m)(e^{-t} - e^{t/2}) \\ R_{2m}[\vec{v}_{2(m-1)}(t)] = v'_{2(m-1)}(t + v_{2(m-1)}(\frac{t}{2}) + v_{1(m-1)}(t) + v_{2(m-1)}(t) - (1-\chi_m)(e^{t} + e^{t/2}), \end{cases} \quad (20)$$

Eq. (19) reduces to

$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - L^{-1}R_{1m}[\vec{v}_{1(m-1)}(t)] \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - L^{-1}R_{2m}[\vec{v}_{2(m-1)}(t)], \end{cases}$$

$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - \int_0^t v'_{1(m-1)}(\tau) - v_{1(m-1)}(\frac{\tau}{2}) - v_{1(m-1)}(\tau) + v_{2(m-1)}(\tau) - (1-\chi_m)(e^{-\tau} - e^{\tau/2})d\tau \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - \int_0^t v'_{2(m-1)}(\tau) + v_{2(m-1)}(\frac{\tau}{2}) + v_{1(m-1)}(\tau) + v_{2(m-1)}(\tau) - (1-\chi_m)(e^{\tau} + e^{\tau/2})d\tau, \end{cases} \quad (21)$$

where

$$\chi_m = \begin{cases} 0 & \text{if } m \le 1 \\ 1 & \text{if } m \ge 2 \end{cases} \quad (22)$$

Consequently,

$$\begin{cases} v_{10}(t) = e^t \\ v_{20}(t) = e^t, \end{cases}$$
$$\begin{cases} v_{11}(t) = -2e^t \\ v_{21}(t) = -2e^t + 2, \end{cases} \quad (23)$$
$$\begin{cases} v_{12}(t) = 2 - 2t - 4e^{t/2} \\ v_{22}(t) = -8 - 2t + 4e^t + 4e^{t/2}, \end{cases}$$
$$\vdots,$$

and the solution is

$$\begin{cases} v_1(t) = v_{10}(t) + \sum_{m=1}^{\infty} v_{1m}(t) \\ \quad = e^t - 2e^t + 2 - 2t - 4e^{t/2} \ldots, \\ v_2(t) = v_{20}(t) + \sum_{m=1}^{\infty} v_{2m}(t) \end{cases} \quad (24)$$
$$= e^t - 2e^t + 2 - 8 - 2t + 4e^t + 4e^{t/2} \ldots$$

There exact solution is

$$\begin{cases} v_1(t) = e^t \\ v_2(t) = e^t. \end{cases} \quad (25)$$

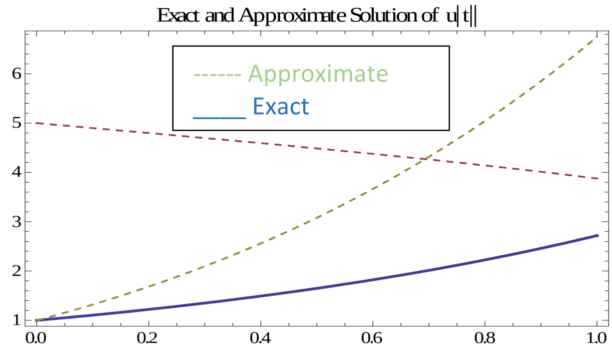

Figure 1: Comparison of Exact and Approximate Solution

Example 3.2 Consider non-linear Delayed system

$$\begin{cases} v_1'(t) = -t\cos\left(\frac{t}{2}\right) + 2v_2\left(\frac{t}{2}\right) + v_3(t) \\ v_2'(t) = 1 - t\sin(t) - 2v_3^{2}\left(\frac{t}{2}\right) \\ v_3'(t) = -t\cos(t) - v_1(t) + v_2(t), \end{cases} \tag{26}$$

initial conditions are

$$\begin{cases} v_1(0) = -1, \\ v_2(0) = 0, \\ v_3(0) = 0. \end{cases} \tag{27}$$

We can choose freely initial approximation $v_0(t)$ to be

$$\begin{cases} v_{10}(t) = -e^{t} \\ v_{20}(t) = t \\ v_{30}(t) = t, \end{cases} \tag{28}$$

Deformation equation of zeros order are

$$\begin{cases} (1-q)L[\phi_1(t;q) - v_{10}(t)] - q\hbar H(t)N[\phi_1(t;q)] = 0 \\ (1-q)L[\phi_2(t;q) - v_{20}(t)] - q\hbar H(t)N[\phi_2(t;q)] = 0 \\ (1-q)L[\phi_3(t;q) - v_{30}(t)] - q\hbar H(t)N[\phi_3(t;q)] = 0, \end{cases} \tag{29}$$

when $q = 0$, Eq. (29) becomes

$$\begin{cases} L[\phi_1(t;q) - v_{10}(t)] = 0 \\ L[\phi_2(t;q) - v_{20}(t)] = 0 \\ L[\phi_3(t;q) - v_{30}(t)] = 0, \end{cases} \tag{30}$$

The initial approximations are

$$\begin{cases} \phi_1(0;q) = v_{10}(0) = -1 \\ \phi_2(0;q) = v_{20}(0) = 0 \\ \phi_3(0;q) = v_{30}(0) = 0, \end{cases} \tag{31}$$

When q=1 nonlinear terms are

$$\begin{cases} N[\phi_1(t;1)] = v_1'(t) + t\cos\left(\frac{t}{2}\right) - 2v_2\left(\frac{t}{2}\right) - v_3(t) \\ N[\phi_2(t;1)] = v_2'(t) - 1 + t\sin(t) + 2v^{2}{}_{3}\left(\frac{t}{2}\right) \\ N[\phi_3(t;1)] = v_2'(t) + t\cos(t) + v_1(t) - v_2(t), \end{cases} \tag{32}$$

Deformation equation of mth order is

$$\begin{cases} L[v_{1m}(t) - \chi_m v_{1(m-1)}(t)] = \hbar H(t)R_{1m}[\vec{v}_{1(m-1)}(t)] \\ L[v_{21m}(t) - \chi_m v_{2(m-1)}(t)] = \hbar H(t)R_{2m}[\vec{v}_{2(m-1)}(t)] \\ L[v_{31m}(t) - \chi_m v_{3(m-1)}(t)] = \hbar H(t)R_{3m}[\vec{v}_{3(m-1)}(t)], \end{cases} \tag{33}$$

By putting $\hbar = -1$, and $H(t) = 1$, in eq. (33), and

$$\begin{cases} R_{1m}\left[\vec{v}_{1(m-1)}(t)\right] = v'_{1(m-1)}(t) - 2v_{2(m-1)}\left(\frac{t}{2}\right) - v_{3(m-1)}(t) + (1-\chi_m)t\cos\left(\frac{t}{2}\right) \\ R_{2m}\left[\vec{v}_{2(m-1)}(t)\right] = v'_{2(m-1)}(t) + 2v^2_{3(m-1)}\left(\frac{t}{2}\right) + (1-\chi_m)(-1 + t\sin(t)) \\ R_{3m}\left[\vec{v}_{3(m-1)}(t)\right] = v'_{3(m-1)}(t) + v_{1(m-1)}(t) - v_{2(m-1)}(t) + (1-\chi_m)(t\cos(t)), \end{cases} \tag{34}$$

Eq. (33) reduces to
$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - L^{-1}R_{1m}\left[\vec{v}_{1(m-1)}(t)\right] \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - L^{-1}R_{2m}\left[\vec{v}_{2(m-1)}(t)\right] \\ v_{3m}(t) = \chi_m v_{3(m-1)}(t) - L^{-1}R_{3m}\left[\vec{v}_{3(m-1)}(t)\right], \end{cases}$$

$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - \int_0^t v'_{1(m-1)}(\tau) - 2v_{2(m-1)}\left(\frac{\tau}{2}\right) - v_{3(m-1)}(\tau) + (1-\chi_m)\tau\cos\left(\frac{\tau}{2}\right)d\tau \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - \int_0^t v'_{2(m-1)}(\tau) + 2v^2_{3(m-1)}\left(\frac{\tau}{2}\right) + (1-\chi_m)(-1 + \tau\sin(\tau))d\tau \\ v_{3m}(t) = \chi_m v_{3(m-1)}(t) - \int_0^t v'_{3(m-1)}(\tau) + v_{1(m-1)}(\tau) - v_{2(m-1)}(\tau) + (1-\chi_m)(\tau\cos(\tau))d\tau, \end{cases} \tag{35}$$

where

$$\chi_m = \begin{cases} 0 & \text{if } m \le 1 \\ 1 & \text{if } m \ge 2 \end{cases} \tag{36}$$

Consequently,

$$\begin{cases} v_{10}(t) = -e^t \\ v_{20}(t) = t \\ v_{30}(t) = t, \end{cases} \tag{37}$$

$$\begin{cases} v_{11}(t) = e^t - 5 + t^2 + 2t\sin\left(\frac{t}{2}\right) + 4\cos\left(\frac{t}{2}\right) \\ v_{21}(t) = t\cos(t) - \sin(t) + \frac{t^3}{3!} \\ v_{31}(t) = -t - t\sin(t) - \cos(t) + \frac{t^2}{2!}, \end{cases} \tag{38}$$

$$\begin{cases} v_{12}(t) = -8 - \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + t\cos(t) - 2\sin(t) + 2t\sin\left(\frac{t}{2}\right) + 4\cos\left(\frac{t}{2}\right) \\ v_{22}(t) = 3t\sin\left(\frac{t}{2}\right) + 6\cos\left(\frac{t}{2}\right) - \frac{t^2}{2!}\cos\left(\frac{t}{2}\right) - \frac{t^3}{12} - \frac{t^4}{64} \\ v_{32}(t) = -2 + 5t + t\sin(t) + 2\cos(t) - e^t - \frac{t^3}{3!} + \frac{t^4}{4!} - 4t\cos\left(\frac{t}{2}\right), \\ \vdots, \end{cases} \tag{39}$$

and the solution is

$$\begin{cases} v_1(t) = v_{10}(t) + \sum_{m=1}^{\infty} v_{1m}(t) \\ \quad = -\cos(t) \\ v_2(t) = v_{20}(t) + \sum_{m=1}^{\infty} v_{2m}(t) \\ \quad = t\cos(t) \\ v_3(t) = v_{30}(t) + \sum_{m=1}^{\infty} v_{3m}(t) \\ \quad = \sin(t). \end{cases} \tag{40}$$

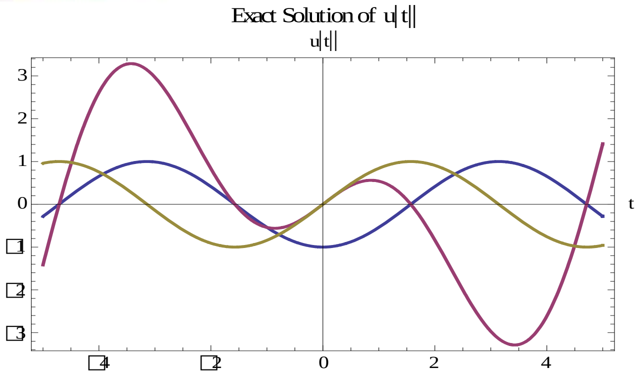

Figure 2: Exact Solution

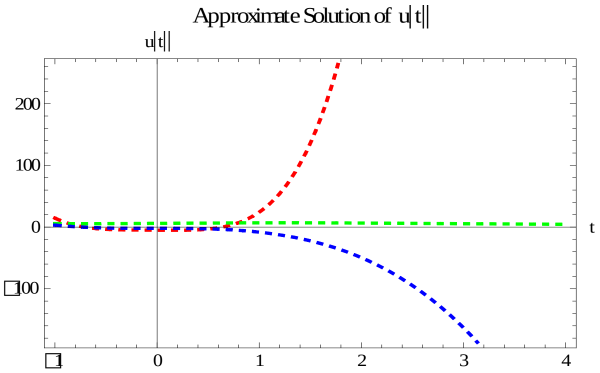

Figure 3: Approximate Solution

Example 3.3 Consider Delay differential system

$$\begin{cases} v_1'(t) = v_1(t-1) \\ v_2'(t) = v_1(t-1) + v_2(t-0.2) \\ v_3'(t) = v_2(t), \end{cases} \tag{41}$$

with initial conditions

$$\begin{cases} v_1(0) = 1, \\ v_2(0) = 1, \\ v_3(0) = 1. \end{cases} \tag{42}$$

We can choose freely initial approximation $v_0(t)$

$$\begin{cases} v_{10}(t) = 1 \\ v_{20}(t) = 1 \\ v_{30}(t) = 1, \end{cases} \tag{43}$$

Deformation equation of zeroth order

$$\begin{cases} (1-q)L[\phi_1(t;q) - v_{10}(t)] - q\hbar H(t)N[\phi_1(t;q)] = 0 \\ (1-q)L[\phi_2(t;q) - v_{20}(t)] - q\hbar H(t)N[\phi_2(t;q)] = 0 \\ (1-q)L[\phi_3(t;q) - v_{30}(t)] - q\hbar H(t)N[\phi_3(t;q)] = 0, \end{cases} \tag{44}$$

When $q = 0$, Eq. (44) becomes

$$\begin{cases} L[\phi_1(t;q) - v_{10}(t)] = 0 \\ L[\phi_2(t;q) - v_{20}(t)] = 0 \\ L[\phi_3(t;q) - v_{30}(t)] = 0, \end{cases} \tag{45}$$

It gives initial approximation

$$\begin{cases} \phi_1(0;q) = v_{10}(0) = 1 \\ \phi_2(0;q) = v_{20}(0) = 1 \\ \phi_3(0;q) = v_{30}(0) = 1, \end{cases} \tag{46}$$

when q=1 nonlinear terms are

$$\begin{cases} N[\phi_1(t;1)] = v_1'(t) - v_1(t-1) \\ N[\phi_2(t;1)] = v_2'(t) - v_1(t-1) - v_2(t-0.2) \\ N[\phi_3(t;1)] = v_2'(t) - v_2(t), \end{cases} \tag{47}$$

Deformation equation of mth order is

$$\begin{cases} L[v_{1m}(t) - \chi_m v_{1(m-1)}(t)] = \hbar H(t) R_{1m}[\vec{v}_{1(m-1)}(t)] \\ L[v_{21m}(t) - \chi_m v_{2(m-1)}(t)] = \hbar H(t) R_{2m}[\vec{v}_{2(m-1)}(t)] \\ L[v_{31m}(t) - \chi_m v_{3(m-1)}(t)] = \hbar H(t) R_{3m}[\vec{v}_{3(m-1)}(t)], \end{cases} \tag{48}$$

By putting $\hbar = -1$, and $H(t) = 1$, in eq. (48), and

$$\begin{cases} R_{1m}\left[\vec{v}_{1(m-1)}(t)\right] = v'_{1(m-1)}(t) - v_{1(m-1)}(t-1) \\ R_{2m}\left[\vec{v}_{2(m-1)}(t)\right] = v'_{2(m-1)}(t) - v_{1(m-1)}(t-1) - v_{2(m-1)}(t-0.2) \\ R_{3m}\left[\vec{v}_{3(m-1)}(t)\right] = v'_{3(m-1)}(t) - v_{2(m-1)}(t), \end{cases} \tag{49}$$

Eq. (48) reduces to

$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - L^{-1}R_{1m}\left[\vec{v}_{1(m-1)}(t)\right] \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - L^{-1}R_{2m}\left[\vec{v}_{2(m-1)}(t)\right] \\ v_{3m}(t) = \chi_m v_{3(m-1)}(t) - L^{-1}R_{3m}\left[\vec{v}_{3(m-1)}(t)\right], \end{cases}$$

$$\begin{cases} v_{1m}(t) = \chi_m v_{1(m-1)}(t) - \int_0^t v'_{1(m-1)}(\tau) - v_{1(m-1)}(\tau-1)d\tau \\ v_{2m}(t) = \chi_m v_{2(m-1)}(t) - \int_0^t v'_{2(m-1)}(\tau) - v_{1(m-1)}(\tau-1) - v_{2(m-1)}(\tau-0.2)d\tau \\ v_{3m}(t) = \chi_m v_{3(m-1)}(t) - \int_0^t v'_{3(m-1)}(\tau) - v_{2(m-1)}(\tau)d\tau, \end{cases} \tag{50}$$

where

$$\chi_m = \begin{cases} 0 & \text{if } m \le 1 \\ 1 & \text{if } m \ge 2 \end{cases}. \tag{51}$$

It gives

$$\begin{cases} v_{10}(t) = 1 \\ v_{20}(t) = 1 \\ v_{30}(t) = 1, \end{cases} \tag{52}$$

$$\begin{cases} v_{11}(t) = t \\ v_{21}(t) = 2t \\ v_{31}(t) = t, \end{cases} \tag{53}$$

$$\begin{cases} v_{12}(t) = -t - \dfrac{t^2}{2!} \\ v_{22}(t) = -1.4t + \dfrac{3t^3}{2} \\ v_{32}(t) = t^2, \end{cases} \tag{54}$$

$$\vdots,$$

and the solution is

$$\begin{cases} v_1(t) = v_{10}(t) + \sum_{m=1}^{\infty} v_{1m}(t) \\ \qquad = e^{t/2}, \\ v_2(t) = v_{20}(t) + \sum_{m=1}^{\infty} v_{2m}(t) \\ \qquad = e^{2t}, \\ v_3(t) = v_{30}(t) + \sum_{m=1}^{\infty} v_{3m}(t) \\ \qquad = \dfrac{1}{2} + \dfrac{1}{2}e^{2t}. \end{cases} \tag{55}$$

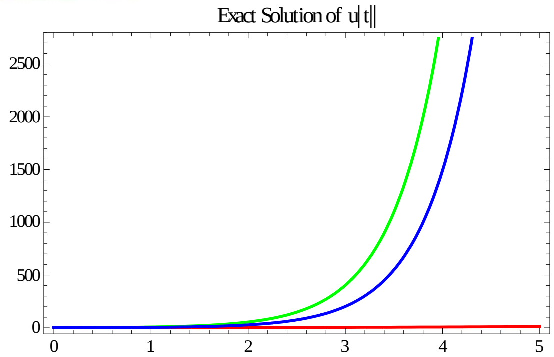

Figure 4: Exact solution

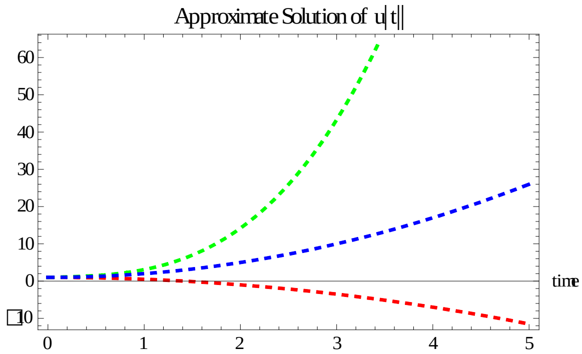

Figure 5: Approximate solution

4. CONCLUSION

In this work, we have worked on delay differential systems via Homotopy Analysis Method. Three numerical examples are solved by using HAM with good approximation. The results obtained by this method provide us quick solution. The comparison of results indicated that the method is extremely effective for system of nonlinear problems. This work illustrates great prospective of the technique for system of nonlinear phenomenon occurring in different fields of science and engineering.

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