ANNALS of Faculty Engineering Hunedoara
– International Journal of Engineering
Tome XV [2017] – Fascicule 3 [August]

ISSN: 1584-2665 [print; online]
ISSN: 1584-2673 [CD-Rom; online]
a free-access multidisciplinary publication
of the Faculty of Engineering Hunedoara

[1.] M. G. SOBAMOWO

ON THE EXPLICIT ANALYTICAL SOLUTIONS TO LARGE AMPLITUDE NONLINEAR OSCILLATIONS ARISING STRUCTURAL ENGINEERING PROBLEMS

[1.]Department of Mechanical Engineering, University of Lagos, Akoka, Lagos, NIGERIA

Abstract: In this work, analytical solutions to large amplitude nonlinear oscillation systems are provided using term by term series integration method. The developed analytical solutions are shown to be valid for both small and large amplitudes of oscillation. The accuracy and explicitness of the analytical solutions were carried out to establish the validity of the method. In conclusion, good agreements are established. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problems than pure numerical or computation methods.
Keywords: Nonlinear; Explicit Analytical Solutions; Large amplitude; Oscillation

1. INTRODUCTION

The modelling of large amplitude of structures such as slender, flexible cantilever beam carrying a lumped mass with rotary inertia at intermediate point along its span and also in fluid-structure interaction resting on nonlinear elastic foundations or subjected to stretching effects often lead to strongly nonlinear models which are not amendable to exact analytical methods. In the cases where the exact analytical solutions are presented either in implicit or explicit form, it involves complex mathematical analysis with possession of high skills in mathematics. Also, such solutions do not provide general exact analytical solutions since they often come with conditional statements which make them limited in used. Application of analytical methods such as Exp-function method, He's Exp-function method, improved F-expansion method, Lindstedt-Poincare techniques, quotient trigonometric function expansion method to the nonlinear equation present analytical solutions either in implicit or explicit form which often involved complex mathematical analysis leading to analytic expression involving a large number terms. Furthermore, the methods are time-consuming task accompanied with possessing high skills in mathematics. Also, they do not provide general analytical solutions since the solutions often come with conditional statements (i.e. except in limited circumstances where exact analytical solutions are possible) which make them limited in used as many of the conditions with the exact solutions do not meet up with the practical applications since they give approximated solutions that hardly provide an all-encompassing understanding of the nature of systems in response to parameters affecting nonlinearity. Also, in practice, analytical solutions with large number of terms and conditional statements for the solutions are not convenient for use by designers and engineers [1]. Consequently, recourse has always been made to numerical methods or approximate analytical methods in solving the problems [2-25]. However, the classical way for finding exact analytical solution is obviously still very important since it serves as an accurate benchmark for numerical solutions. Also, the experimental data are useful to access the mathematical models, but are never sufficient to verify the numerical solutions of the established mathematical models. Comparison between the numerical calculations and experimental data often fail to reveal the compensation of modelling deficiencies through the computational errors or unconscious approximations in establishing applicable numerical schemes. Additionally, exact analytical solutions for specified problems are also essential for the development of efficient applied numerical simulation tools. Inevitably, exact analytical expressions are required to show the direct relationship between the models parameters. When such exact analytical solutions are available, they provide good insights into the significance of various system parameters affecting the phenomena as it gives continuous physical insights than pure numerical or computation methods. Furthermore, most of the analytical approximation and purely numerical methods that were applied in literatures to nonlinear problems are computationally intensive. Exact analytical expression is more convenient for engineering calculations compare with experimental or numerical studies and it is obvious starting point for a better understanding of the relationship between physical quantities/properties. It is convenient for parametric studies, accounting for the physics of the problem and appears more appealing than the numerical solutions. It appears more appealing than the numerical solution as it helps to reduce the computation costs, simulations and task in the analysis of real life problems. Therefore, an exact analytical solution is required for the problem. Therefore, in this research, analytical solutions are provided to the nonlinear Duffing Oscillators. Simplicity, flexibility in application, and avoidance of complicated numerical integration are some of the added advantages over the previous methods. It provides complementary advantages of higher accuracy, reduced computation cost and task as compared to the other methods as found in literatures. The developed analytical solutions are compared with the numerical results and the results of approximate analytical solutions and good agreements reached. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problem than pure numerical or computation methods.

2. DEVELOPMENT OF EXACT ANALYTICAL SOLUTIONS FOR THE NATURAL FREQUENCIES OF THE STRUCTURES

$$\ddot{u}(\tau)+\alpha u(\tau)+\beta u^3(\tau)=0 \tag{1}$$

The initial conditions are

$$u(0)=A\ \dot{u}(0)=0$$

Integrating Eq. (1), we have

$$\frac{1}{2}\dot{u}^2(\tau)+\frac{\alpha}{2}u^2(\tau)+\frac{\beta}{4}u^4(\tau)=c \tag{2}$$

where c is a constant. On imposing the initial condition, we have

$$c=\frac{\alpha}{2}A^2+\frac{\beta}{4}A^4 \tag{3}$$

Substituting Eq. (3) into Eq. (2), we have

$$\frac{1}{2}\dot{u}^2(\tau)+\frac{\alpha}{2}u^2(\tau)+\frac{\beta}{4}u^4(\tau)=\frac{\alpha}{2}A^2+\frac{\beta}{4}A^4 \tag{4}$$

which gives

$$dt=\frac{du}{\sqrt{\alpha(A^2-u^2)+\frac{\beta}{2}(A^4-u^4)}} \tag{5}$$

Integrating Eq. (5)

$$\int_0^{\frac{T_p}{4}}dt=\int_0^A\frac{du}{\sqrt{\alpha(A^2-u^2)+\frac{\beta}{2}(A^4-u^4)}} \tag{6}$$

Then we

$$T_p(A)=4\int_0^A\frac{du}{\sqrt{\alpha(A^2-u^2)+\frac{\beta}{2}(A^4-u^4)}} \tag{7}$$

On substituting, $u = Asint$, we have

$$T_p=4\int_0^{\frac{\pi}{2}}\frac{A\cos t dt}{\sqrt{\alpha A^2(1-sin^2t)+\frac{\beta}{2}A^4(1-sin^4t)}} \tag{8}$$

which gives

$$T_p=4\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{\left[\alpha+\frac{\beta A^2}{2}\right]+\frac{\beta A^2}{2}sin^2t}} \tag{9}$$

and

$$T_p = 4\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\left[\alpha+\frac{\beta A^2}{2}\right]}\sqrt{1-\left\{\frac{-\frac{\beta A^2}{2}}{\left(\alpha+\frac{\beta A^2}{2}\right)}\right\}sin^2 t}} \tag{10}$$

which gives

$$T_p = \frac{4}{\xi_1}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\xi_2^2 sin^2 t}} \tag{11}$$

where

$$\xi_1 = \sqrt{\left[\alpha+\frac{\beta A^2}{2}\right]} \qquad \xi_2 = \frac{-\frac{\beta A^2}{2}}{\left(\alpha+\frac{\beta A^2}{2}\right)}$$

The above Eq. (11) is called the complete elliptic integral of first kind

$$T_p = \frac{4}{\xi_1}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\xi_2^2 sin^2 t}} \qquad \xi_2^2 < 1 \tag{12}$$

In order to evaluate the integral, we expand the integral in the form of series as

$$\frac{1}{\sqrt{1-\xi_2^2 sih^2 t}} = 1+\frac{\xi_2^2}{2}sin^2 t+\frac{3\xi_2^2}{8}sin^4 t+\frac{5\xi_2^6}{16}sin^6 t+\frac{35\xi_2^8}{128}sin^8 t+\ldots \tag{13}$$

The above series is uniformly convergent for all ξ_2, and may, therefore, be integrated term by term. Then, we have

$$T_p = \frac{4}{\xi_1}\int_0^{\frac{\pi}{2}}\left(1+\frac{\xi_2^2}{2}sin^2 t+\frac{3\xi_2^4}{8}sin^4 t+\frac{5\xi_2^6}{16}sin^6 t+\frac{35\xi_2^8}{128}sin^8 t+\ldots\right)dt \tag{14}$$

After integration

$$T_p = \frac{2\pi}{\xi_1}+\frac{2\pi}{\xi_1}\left(\left(\frac{1}{2}\right)^2\xi_2^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\xi_2^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\xi_2^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\xi_2^8+\ldots+\left(\prod_{n=1}^{N}\frac{2n-1}{2n}\right)^2\xi_2^{2N}\right) \tag{15}$$

But $T_p(A) = \frac{2\pi}{\omega} \rightarrow \omega = \frac{2\pi}{T_p(A)}$

$$\omega = \frac{\xi_1}{1+\left(\left(\frac{1}{2}\right)^2\xi_2^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\xi_2^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\xi_2^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\xi_2^8+\ldots+\left(\prod_{n=1}^{N}\frac{2n-1}{2n}\right)^2\xi_2^{2N}\right)} \tag{16}$$

It can easily be seen that as the nonlinear term tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{\xi_2\to 0}\frac{\omega}{\omega_b} = 1 \tag{17}$$

Also, as the amplitude A tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{A\to 0}\frac{\omega}{\omega_b} = 1 \tag{18}$$

For very large values of the amplitude A, we have

$$\lim_{A\to \infty}\frac{\omega}{\omega_b} = \infty \tag{19}$$

Alternatively, we can have

$$\omega_{1,exact} = \frac{\pi\xi_1}{2\left[\int_0^{\pi/2}\left(1-\xi_2 sin^2 t\right)^{-\frac{1}{2}}dt\right]} \tag{20}$$

Using term-by-term series integration method, we developed an approximation analytical solution for the nonlinear natural frequency as

$$\omega_1 = \frac{\xi_1}{\left(1+0.25000\xi_2+0.11680\xi_2^{\ 2}+0.59601\xi_2^{\ 3}-1.90478\xi_2^{\ 4}+2.04574\xi_2^{\ 5}\right)} \tag{21}$$

The ratio of the nonlinear frequency, ω_1 to the linear frequency, ω_b

$$\frac{\omega_1}{\omega_b} = \frac{1}{\left(1+0.25000\xi_2+0.11680\xi_2^{\ 2}+0.59601\xi_2^{\ 3}-1.90478\xi_2^{\ 4}+2.04574\xi_2^{\ 5}\right)} \tag{22}$$

Also, it can easily be seen that as the nonlinear term tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{\xi_2\to 0}\frac{\omega}{\omega_b}=1 \tag{23}$$

Also, as the amplitude A tends to zero, the frequency ratio of the nonlinear frequency to the linear frequency, $\frac{\omega}{\omega_b}$ tends to 1.

$$\lim_{A\to 0}\frac{\omega}{\omega_b}=1 \tag{24}$$

Also, it should be pointed out that when the nonlinear term, V is set to zero, we recovered the linear natural frequency

$$\omega_b = \sqrt{\left[\frac{(K+C)}{M}\right]} \tag{25}$$

3. DEVELOPMENT OF ANALYTICAL SOLUTIONS FOR THE DYNAMIC RESPONSE OF STRUCTURES

$$dt = \frac{du}{\sqrt{\frac{(K+C)}{M}(A^2-u^2)-\frac{V}{2M}(A^4-u^4)}} \tag{26}$$

which gives

$$dt = \frac{du}{\sqrt{\frac{(K+C)}{M}(A^2-u^2)-\frac{V}{2M}(A^4-u^4)}} \tag{27}$$

Integrating Eq. (27)

$$\int dt = \int\frac{du}{\sqrt{\frac{(K+C)}{M}(A^2-u^2)-\frac{V}{2M}(A^4-u^4)}} \tag{28}$$

On substituting $u = Asin\varphi$, we have

$$t = \int\frac{d\varphi}{\xi_1\sqrt{1-\xi_2^2 sin^2\varphi}} \tag{29}$$

In order to evaluate the integral, we expand the integral in the form

$$\frac{1}{\sqrt{1-\xi_2^2 sin^2\varphi}} = 1+\frac{\xi_2^2}{2}sin^2\varphi+\frac{3\xi_2^2}{8}sin^4\varphi+\frac{5\xi_2^6}{16}sin^6\varphi+\ldots \tag{30}$$

The above series is uniformly convergent for all ξ_2, and may, therefore, be integrated term by term. Then, we have

$$t = \frac{1}{\xi_1}\int\left(1+\frac{\xi_2^2}{2}sin^2\varphi+\frac{3\xi_2^4}{8}sin^4\varphi+\frac{5\xi_2^6}{16}sin^6\varphi+\frac{35\xi_2^8}{128}sin^8\varphi+\ldots\right)d\varphi \tag{31}$$

Where

$$\int sin^{2m}\varphi d\varphi = \frac{-cos\varphi}{2m}\left\{sin^{2m-1}\varphi+\sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}sin^{2m-2k-1}\varphi\right\}+\frac{(2m-1)!!}{2^m m!}\varphi \tag{32}$$

Alternatively, by series expansion, we have

$$\frac{1}{\sqrt{1-\xi_2^2 sin^2\varphi}} = 1+\frac{\xi_2^2}{2}\varphi^2+\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\varphi^4+\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\varphi^6+\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\varphi^8+\ldots \tag{33}$$

Substitute Eq. (33) into Eq.(29), we have

$$t=\frac{1}{\xi_1}\int\left\{1+\frac{\xi_2^2}{2}\varphi^2+\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\varphi^4+\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\varphi^6+\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\varphi^8+\ldots\right\}d\varphi \tag{34}$$

$$t=\frac{1}{\xi_1}\left\{\begin{array}{l}\varphi+\frac{\xi_2^2}{6}\varphi^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\varphi^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\varphi^7\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\varphi^9+\ldots\end{array}\right\}+\Phi \tag{35}$$

Applying, the second initial condition

$$\Phi=-\frac{1}{\xi_1}\left\{\begin{array}{l}A+\frac{\xi_2^2}{6}A^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)A^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)A^7\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)A^9+\ldots\end{array}\right\} \tag{36}$$

Substitute Eq. (35) into Eq. (36), we have

$$t=\frac{1}{\xi_1}\left\{\left\{\begin{array}{l}\varphi+\frac{\xi_2^2}{6}\varphi^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\varphi^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\varphi^7\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\varphi^9+\ldots\\-\left\{\begin{array}{l}A+\frac{\xi_2^2}{6}A^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)A^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)A^7\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)A^9+\ldots\end{array}\right\}\end{array}\right\}\right\} \tag{37}$$

Which could be written as

$$t=\frac{1}{\xi_1}\left\{\begin{array}{l}(\varphi-A)+\frac{\xi_2^2}{6}\left(\varphi^3-A^3\right)+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\left(\varphi^5-A^5\right)+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\left(\varphi^7-A^7\right)\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\left(\varphi^9-A^9\right)+\ldots\end{array}\right\} \tag{38}$$

$$t\xi_1-\Phi=\varphi+\frac{\xi_2^2}{6}\varphi^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)\varphi^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)\varphi^7+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)\varphi^9 \tag{39}$$

By reversion of power series,

$$\begin{array}{l}\varphi=\lambda_1\left(t\xi_1-\Phi\right)+\lambda_2\left(t\xi_1-\Phi\right)^2+\lambda_3\left(t\xi_1-\Phi\right)^3+\lambda_4\left(t\xi_1-\Phi\right)^4\\+\lambda_5\left(t\xi_1-\Phi\right)^5+\lambda_6\left(t\xi_1-\Phi\right)^6+\lambda_7\left(t\xi_1-\Phi\right)^7+\ldots\end{array} \tag{40}$$

where $\lambda_1=\frac{1}{\beta_1}$ $\lambda_2=-\frac{\beta_2}{\beta_1^3}$ $\lambda_3=\frac{2\beta_2^2-\beta_1\beta_3}{\beta_1^5}$ $\lambda_5=\frac{6\beta_1^2\beta_2\beta_4+3\beta_1^2\beta_3^2-\beta_1^3\beta_5+14\beta_2^4-21\beta_1\beta_2^2\beta_3}{\beta_1^9}$

$$\lambda_4=\frac{5\beta_1\beta_2\beta_3-5\beta_2^3-\beta_1^2\beta_4}{\beta_1^7}\quad \lambda_6=\frac{7\beta_1^3\beta_2\beta_5+84\beta_1\beta_2^3\beta_3+7\beta_1^3\beta_3\beta_4-28\beta_1^2\beta_2\beta_3^2-\beta_1^4\beta_6-28\beta_1^2\beta_2^2\beta_4-42\beta_2^5}{\beta_1^{11}}$$

$$\lambda_7=\frac{\begin{array}{l}\{8\beta_1^4\beta_2\beta_6+8\beta_1^4\beta_3\beta_5+4\beta_1^4\beta_4^2+120\beta_1^2\beta_2^3\beta_4+180\beta_1^2\beta_2^2\beta_3^2+132\beta_2^6-\beta_1^5\beta_7-36\beta_1^3\beta_2^2\beta_5\\-72\beta_1^3\beta_2\beta_3\beta_4-12\beta_1^3\beta_3^3-330\beta_1\beta_2^4\beta_3\}\end{array}}{\beta_1^{13}}$$

where $\beta_1=1$ $\beta_2=\beta_4=\beta_6=0$ $\beta_3=\frac{\xi_2^2}{6}$ $\beta_3=\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)$ $\beta_7=\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)$ $\beta_9=\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)$

Recall that $u=A sin\varphi$. Substituting Eq. (81), we have

$$u(t)=A sin\left\{\begin{array}{l}\lambda_1\left(t\xi_1-\Phi\right)+\lambda_2\left(t\xi_1-\Phi\right)^2+\lambda_3\left(t\xi_1-\Phi\right)^3+\lambda_4\left(t\xi_1-\Phi\right)^4\\+\lambda_5\left(t\xi_1-\Phi\right)^5+\lambda_6\left(t\xi_1-\Phi\right)^6+\lambda_7\left(t\xi_1-\Phi\right)^7\end{array}\right\} \tag{41}$$

It can easily be shown that

$$\lim_{t\to 0}u(t)=A \tag{42}$$

$$sin\left\{\begin{array}{l}\lambda_1(-\Phi)+\lambda_2(-\Phi)^2+\lambda_3(-\Phi)^3+\lambda_4(-\Phi)^4\\+\lambda_5(-\Phi)^5+\lambda_6(-\Phi)^6+\lambda_7(-\Phi)^7\end{array}\right\}\to 1$$ where $$-\Phi=\frac{1}{\xi_1}\left\{\begin{array}{l}A+\frac{\xi_2^2}{6}A^3+\frac{1}{5}\left(\frac{\xi_2^2}{6}+\frac{3\xi_2^4}{8}\right)A^5+\frac{1}{7}\left(\frac{\xi_2^2}{45}-\frac{\xi_2^4}{4}+\frac{5\xi_2^6}{16}\right)A^7\\+\frac{1}{9}\left(\frac{\xi_2^8}{128}-\frac{5\xi_2^6}{16}-\frac{3\xi_2^4}{40}-\frac{\xi_2^2}{630}\right)A^9+\ldots\end{array}\right\}$$

4. CONCLUSION

Analytical solutions to large amplitude nonlinear oscillation systems have been provided. The developed analytical solutions are shown to be valid for both small and large amplitudes of oscillation. The accuracy and explicitness of the analytical solutions were carried out to establish the validity of the method. In conclusion, good agreements are established. The analytical solutions can serve as a starting point for a better understanding of the relationship between the physical quantities of the problems as it provides continuous physical insights into the problems than pure numerical or computation methods.

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