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AN EFFICIİENT TECHNIQUE TO DETERMINE THE TEMPERATURE DISTRIBUTION WITHIN ANNULAR FINS

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Abstract: In this paper, the homotopy perturbation method (HPM) is used to determine the temperature distribution within annular fins with temperature-dependent thermal conductivity and a step reduction in thickness towards the fin tip.This approach is quite simple and easy to use. The solution obtained by this approach is in the form of a convergent series with an easily computable component. This approach does not need require any assumption like weak nonlinearity assumptions, linearization, or perturbation theory. The high accuracy is obtained by the suggested algorithm moreover, computational results shows that the HPM is efficient, accurate and easy to use.
Keywords: temperature distribution, annular fin, homotopy perturbation method, variable conductivity

1. INTRODUCTION

Many physics and engineering problems are modeled by partial differential equations. In many instances these equations are nonlinear and exact solutions are very difficult to obtain. Numerical methods were developed in order to find approximate solutions to these nonlinear equations. However, numerical solutions are insufficient to determine general properties of certain systems of equations and thus analytical methods have been developed.

Homotopy perturbation method was originally introduced by He [1-3]. It is on the base of homotopy technique and traditional perturbation method. In most cases a quick convergent series can get by this method. For numerical purpose with a high degree of accuracy usually a few number of term of the series can be used. This method is effective in partial differential equations in boundary value problems and in different fields. The numerical solution of a equation can obtain without discretization, it is not effected by mulitiple rounding errors and do not require large computer memory and time.

In addition there is no general procedure which is applicable for all such equations. So each equation has to be studied by considering as an individual problem. For this goal, many novel methods for the detection of exact travelling wave solutions of NLEEs have been drawn in huge combinations by a large number of experts. As a result, a lot of work has been done in formulation of several convincing and significant techniques. Different researchers apply these techniques on mathematical and physical models. Such as, the homogenous balance method (Wang, 1995; Zayed, Zedan and Gepreel, 2004), Hirota's bilinear transformation method (Hirota, 1973; Hirota and Satsuma, 1981), auxiliary equation method (Sirendaoreji, 2004), trial function method (Inc and Evans, 2004), Jacobi elliptic function method (Ali, 2011), tanh-function method (Abdou, 2007; Fan, 2000; Malfliet, 1992), homotopy perturbation method (Mohyud-Din et al., 2011), sine-cosine method (Wazwaz, 2004; Bibi and S.T.Mohyud-Din, 2013), truncated Painleve expansion method (Weiss et al., 1983), variational iteration method (He, 1997; Abdou and Soliman, 2005; Abbasbandy, 2007), Exp-function method (He and Wu, 2006; Akbar and Ali, 2012; Naher et al., 2012), (G'/G)-expansion method (Wang et al., 2008; Akbar et al., 2012; Zayed, 2010; Zayed and Gepreel, 2009;Ali, 2011; Zayed, 2009; Shehata, 2010), improved

(G'/G)-expansion method (Zhang, F et al., 2010), exp(-φ(ξ))-expansion method (Khan and Akbar, 2013) and so on.

He [1-3] proposed a perturbation technique, so called He's homotopy perturbation method (HPM). This method does not require a small parameter in the equation and takes the full advantage of the traditional homotopy techniques and perturbation methods. Relatively recent survey on the method and its applications can be found in Refs. [4-22].

This paper deals with the concentric annular fins with a step change in thickness and temperature-dependent thermal conductivity. Finally two nonlinear heat transfer equations with nonlinear boundary conditions are obtaine which are then solved by homotopy perturbation method. Recently, Aslantürk [23] used Adomian decomposition method for solving the governing problem.

2. HOMOTOPY PERTURBATION METHOD HPM PRINCIPLE

For a general nonlinear boundary-conditioned differential equation

$$A(u) - f(r) = 0, r \in \Omega \tag{1}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, u is the r-dependent unknown function is the boundary of the domain Ω.

The operator A can, generally speaking, be divided in to two parts L and N, where L is linear, and N is nonlinear, therefore

Eq. (8) can be written as

$$L(u) + N(u) - f(r) = 0 \tag{2}$$

By using the homotopy technique, one can construct a homotopy $V(r,p): \Omega \times [0,1] \to A$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{3}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[A(v) - f(r)] = 0 \tag{4}$$

Where $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation which satisfies the boundary conditions

Clearly, we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{5}$$

And, we also have

$$H(v,1) = A(v) - f(r) = 0 \tag{6}$$

The changing process of p from zero to unity is just that of $V(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If the embedding parameter p; $(0 \le p \le 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of Eq. (3) can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{7}$$

and after setting p = 1 results in the approximate solution of Eq. (1) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots$$

3. PROBLEM DESCRİPTİON

İn the problem we analyze the fin parameters using an annular fin with temperature-dependent thermal conductivity and a step change in thickness as shown in Figure 1. The fin is exposed to a convective environment at the constant ambient temperature T_a and heat transfer coefficient *h*. The base temperature T_b of the fin is constant. Since the fin is supposed to be thin, the temperature distribution within the fin does not depend on axial direction. The energy balance equations are given

$$2t\frac{d}{dr}\left[k(T_1)r\frac{dT_1}{dr}\right] - 2hr(T_1 - T_a) = 0, \quad r_i < r < r_{in} \tag{8}$$

$$2\lambda t\frac{d}{dr}\left[k(T_2)r\frac{dT_2}{dr}\right] - 2hr(T_2 - T_a) = 0, \quad r_{in} < r < r_0 \tag{9}$$

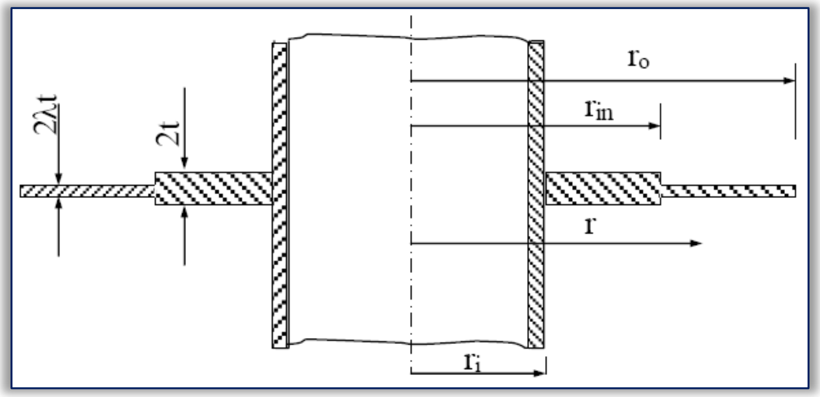

Figure 1. Schematic diagram of an annular fin with a step change in thickness

The thermal conductivity of the fin material is assumed to be a linear function of temperature according to

$$k(T) = k_a\left[1+\gamma(T-T_a)\right] \tag{10}$$

where k_a is the thermal conductivity at the ambient fluid temperature of the fin, κ is the parameter describing the variation of thermal conductivity. Employing the following dimensionless parameters

$$\xi = \frac{r-r_i}{r_i} \quad \eta = \frac{r-r_{in}}{r_{in}} \quad R = \frac{r}{r_i} \quad R_{in} = \frac{r_{in}}{r_i} \quad R_0 = \frac{r_0}{r_i} \tag{11}$$

$$\delta = \frac{t}{r_i} \quad Bi = \frac{hr_i}{k_a} \quad M_1 = \sqrt{\frac{Bi}{\delta}} \quad M_2 = \frac{M_1 R_{in}}{\sqrt{\lambda}}$$

$$\phi(\eta) = \frac{T_2-T_a}{T_b-T_a} \quad \theta(\xi) = \frac{T_1-T_a}{T_b-T_a} \quad \kappa = \gamma(T_b-T_a) \quad U = \delta\left[(1-\lambda)R_{in} + \lambda R_0 - 1\right]$$

the formulation of the problem reduces to

$$\frac{d^2\theta}{\partial\xi^2} = M_1^2\theta - \kappa\theta\frac{d^2\theta}{\partial\xi^2} - \kappa\left(\frac{d\theta}{\partial\xi}\right)^2 - \frac{\kappa}{(1+\xi)}\theta\frac{d\theta}{\partial\xi} - \frac{1}{(1+\xi)}\frac{d\theta}{\partial\xi}, \quad 0<\xi<R_{in}-1 \tag{12}$$

$$\frac{d^2\phi}{\partial\eta^2} = M_2^2\phi - \kappa\phi\frac{d^2\phi}{\partial\eta^2} - \kappa\left(\frac{d\phi}{\partial\eta}\right)^2 - \frac{\kappa}{(1+\eta)}\phi\frac{d\phi}{\partial\eta} - \frac{1}{(1+\eta)}\frac{d\phi}{\partial\eta}, \quad 0<\eta<\frac{R_0}{R_{in}}-1 \tag{13}$$

$$\theta(0) = 1 \tag{14}$$

$$\theta(R_{in}-1) = \phi(0) \tag{15}$$

$$\left[(1+\kappa\theta)\frac{d\theta}{d\xi} + Bi(1-\lambda)\theta\right]_{\xi=R_{in-1}} = \frac{\lambda}{R_{in}}\left[(1+\kappa\phi)\frac{d\phi}{d\eta}\right]_{\eta=0} \tag{16}$$

$$= \left[(1+\kappa\phi)\frac{d\phi}{d\eta} + R_{in}Bi\phi\right]_{\eta=\frac{R_0}{R_{in}}-1} = 0 \tag{17}$$

We construct the following homotopy

$$\frac{d^2\theta}{\partial\xi^2} = p\left\{M_1^2\theta - \kappa\theta\frac{d^2\theta}{\partial\xi^2} - \kappa\left(\frac{d\theta}{\partial\xi}\right)^2 - \frac{\kappa}{(1+\xi)}\theta\frac{d\theta}{\partial\xi} - \frac{1}{(1+\xi)}\frac{d\theta}{\partial\xi}\right\} \tag{18}$$

Assume the solution of Eq. (18) to be in the form:

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \ldots \tag{19}$$

Substituting (19) into (18) and equating the coefficients of like powers of p, we have the following set of differential equations

$$p^0 : \frac{d^2\theta_0}{\partial\xi^2} = 0$$

$$p^1: \frac{d^2\theta_1}{\partial\xi^2} = M_1^2\theta_0 - \kappa\theta_0\frac{d^2\theta_0}{\partial\xi^2} - \kappa\left(\frac{d\theta_0}{\partial\xi}\right)^2 - \frac{\kappa}{(1+\xi)}\theta_0\frac{d\theta_0}{\partial\xi} - \frac{1}{(1+\xi)}\frac{d\theta_0}{\partial\xi} = 0$$

$$p^2: \frac{d^2\theta_2}{\partial\xi^2} = M_1^2\theta_1 - \kappa\left(\theta_1\frac{d^2\theta_0}{\partial\xi^2} + \theta_0\frac{d^2\theta_1}{\partial\xi^2}\right) - \kappa\left(2\frac{d\theta_0}{\partial\xi}\frac{d\theta_1}{\partial\xi}\right) - \frac{\kappa}{(1+\xi)}\left(\theta_1\frac{d\theta_0}{\partial\xi} + \theta_0\frac{d\theta_1}{\partial\xi}\right) - \frac{1}{(1+\xi)}\frac{d\theta_1}{\partial\xi} = 0$$

$$p^3: \frac{d^2\theta_3}{\partial\xi^2} = M_1^2\theta_2 - \kappa\left(\theta_2\frac{d^2\theta_0}{\partial\xi^2} + \theta_1\frac{d^2\theta_1}{\partial\xi^2} + \theta_0\frac{d^2\theta_2}{\partial\xi^2}\right) - \kappa\left(\left(\frac{d\theta_1}{\partial\xi}\right)^2 + 2\frac{d\theta_0}{\partial\xi}\frac{d\theta_2}{\partial\xi}\right)$$

$$-\frac{\kappa}{(1+\xi)}\left(\theta_2\frac{d\theta_0}{\partial\xi} + \theta_1\frac{d\theta_1}{\partial\xi} + \theta_0\frac{d\theta_2}{\partial\xi}\right) - \frac{1}{(1+\xi)}\frac{d\theta_2}{\partial\xi} = 0$$

...

and so on, the rest of the polynomials can be constructed in a similar manner. So we have

$$\sum_0^\infty \theta_m = \theta_0 + M_1^2\int_0^\varepsilon\int_0^\varepsilon\left(\sum_{m=0}^\infty \theta_m\right)d\xi d\xi - \int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{(1+\xi)}\frac{d}{d\xi}\left(\sum_{m=0}^\infty \theta_m\right)\right)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon\left(\sum_{m=0}^\infty A_m\right)d\xi d\xi$$

$$-\kappa\int_0^\varepsilon\int_0^\varepsilon\left(\sum_{m=0}^\infty B_m\right)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{1+\xi}\left(\sum_{m=0}^\infty C_m\right)\right)d\xi d\xi$$

Let $\frac{d\theta(0)}{\partial\xi} = c_1$, together with $\theta(0) = 1$.

$$\theta_0 = 1 + c_1\xi \tag{20}$$

Consequently it can be written following recursive relationship, such as

$$\theta_{m+1} = M_1^2\int_0^\varepsilon\int_0^\varepsilon(\theta_m)d\xi d\xi - \int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{(1+\xi)}\frac{d\theta_m}{d\xi}\right)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon(A_m)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon(B_m)d\xi d\xi$$

$$-\kappa\int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{(1+\xi)}C_m\right)d\xi d\xi \qquad , m \geq 0$$

Letting $\theta(0) = c_2$, $\frac{d\phi(0)}{\partial\eta} = c_3$, and applying the same procedure to Eq. (13), it can be written as follows:

$$\phi_0 = c_2 + c_3\xi$$

$$\phi_{m+1} = M_2^2\int_0^\varepsilon\int_0^\varepsilon(\phi_m)d\xi d\xi - \int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{(1+\xi)}\frac{d\phi_m}{d\xi}\right)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon(A_m)d\xi d\xi - \kappa\int_0^\varepsilon\int_0^\varepsilon(B_m)d\xi d\xi$$

$$-\kappa\int_0^\varepsilon\int_0^\varepsilon\left(\frac{1}{(1+\xi)}C_m\right)d\xi d\xi \qquad , m \geq 0$$

4. HEAT TRANSFER RATE

Total heat transfer rate is expressed by applying the Fourier's law at the fin base, as

$$Q = -4\pi k_b r_i t\left.\frac{dT_1}{dr}\right|_{r=r_i} \tag{21}$$

The dimensionless heat transfer rate is defined as follows:

$$q = \frac{Q}{4\pi k_a r_i (T_b - T_a)} = -(1+\kappa)\delta\left.\frac{d\theta}{d\xi}\right|_{\xi=0} \tag{22}$$

Since the temperature gradient at the fin base equal to the integral constant c_1, the dimensionless heat transfer rate rewritten as

$$q = -(1+\kappa)\delta c_1 \tag{23}$$

Table 1. Comparison Homotopy Perturbation Method and analytical solution for the case of constant thermal conductivity ($R_{in} = 1.5$, $R_0 = 2.0$)

	Bi=0.01				Bi=0.10			
	$\lambda = 0.5, \delta = 0.0047$		$\lambda = 1.0, \delta = 0.0033$		$\lambda = 0.5, \delta = 0.047$		$\lambda = 1.0, \delta = 0.033$	
R	Exact [24]	HPM	Exact [24]	HPM	Exact [24]	HPM	Exact [24]	HPM
1.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1.1	0.854150	0.854158	0.819271	0.819275	0.851624	0.851632	0.817925	0.817930
1.2	0.738402	0.738417	0.677861	0.677869	0.733519	0.733534	0.675249	0.675257
1.3	0.647023	0.647044	0.567358	0.567370	0.639872	0.639893	0.563505	0.563517
1.4	0.575688	0.575713	0.481488	0.481500	0.566291	0.566315	0.476374	0.476386
1.5	0.521115	0.521139	0.415531	0.415542	0.509432	0.509456	0.409096	0.409107
1.6	0.440696	0.440716	0.365931	0.365941	0.428699	0.428719	0.358072	0.358081
1.7	0.383386	0.383403	0.330014	0.330023	0.370598	0.370615	0.320588	0.320597
1.8	0.345241	0.345257	0.305795	0.305803	0.331178	0.331192	0.294616	0.294624
1.9	0.323490	0.323504	0.291836	0.291844	0.307636	0.307649	0.278672	0.278679
2.0	0.316301	0.316314	0.287148	0.287155	0.298089	0.298102	0.271714	0.271721

Nomenclature

h - heat transfer coefficient, [w/m^{2}K]
k - thermal conductivity, [w/mK]
k_a - thermal conductivity at ambient fluid temperature, [w/mK]
M - convectional fin parameter
N - nonlinear operator
r - radial coordinate, [m]
r_{in} - intermediate radius, [m]
r_i - inner radius of annular fin, [m]
r_o- outer radius of annular fin, [m]
R - remainder of the linear operator or dimensionless radial coordinate
R_{in} - the ratio of intermediate radius to inner radius
R_0 - the ratio of outer radius to inner radius
t - unreduced half-thickness of fin, [m]
T - temperature, [K]
U - dimensionless fin volume, $(V/2\pi r_i^3)$
V - fin volume, [m^3]
δ - dimensionless unreduced fin thickness
ξ - dimensionless radial coordinate from the fin base
μ - dimensionless radial coordinate from the reduction radius towards the fin tip
ϒ - the slope of the thermal conductivitytemperature divided by the intercept k_a
Θ - dimensionless temperature
Φ - dimensionless temperature
λ - thickness ratio which describes the variation of fin thickness
κ - a parameter describing variation of thermal conductivity

5. CONCLUSİONS

This paper discussed the homotopy perturbatin method (HPM) to calculate an essentially closed form solution for an annular fin with a step change in thickness and temperature-dependent thermal conductivity. Based on one dimensional conduction two nonlinear differential equations with nonlinear inhomogeneous boundary conditions have been obtained. The homotopy perturbation method has been used to fined out the temperature distribution within the fin. The results reveal that the temperature profile has an abrupt change in the temperature gradient where the step change in thickness occurs. The temperature profile and the heat transfer rate depends on and thermal conductivity parameter.

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