[1.] Narinder Kumar WADHAWAN, [2.]Priyanka WADHAWAN

SUM OF CUBES WHICH EQUALS ANOTHER CUBE, DIOPHANTINE EQUATIONS—THIRD POWER

[1.]Indian Administrative Services, Punjab Engineering College Then Affiliated With Punjab University, Chandigarh, INDIA
[2.]Department of Computer Sciences, Thapar Institute Of Engineering And Technology, Thapar University, Patiala, INDIA

Abstract: Cubes that sum up to another cube have been determined in this research paper. There is no upper limit on the number of adding cubes in integer form, however, lower limit is four in view of Fermat Last theorem. Concept behind the cubes which add up to give another cube, is that these cubes can always be represented by a cubic equation say in variable x and this is then transformed to linear equation by equating coefficient of x square to zero and also after self-cancellation of terms containing x cube. Self-cancellation of x cube terms is feasible only if total number of cubes is even. Since number of terms is not necessarily even, therefore, a formula has been devised for determining equation that helps cancellation of x cube terms. This equation called seed or empirical equation then populates cubes in integer form. Not limiting to determining such cubes, parametrisation that leads to results, has been determined. Research papers starts with basic mathematics and ends up in sophisticated results. Hence, unlike other papers, it is considered easily comprehensible to scholars and students alike.
Keywords: integers, rational quantity, cubic equation, linear, taxicab and cab taxi numbers, diophantine

1. INTRODUCTION

A rational and real quantity say n can always be expressed as

$$n = a.x + A \tag{1}$$

where a, x and A are rational and real quantities which can be assigned infinite rational real values to satisfy $n = a.x + A$. For example, n say 5 can be expressed as $5 = 2.(2) + 1 = 1.(2) + 3 = 10 - 5.(1) = \frac{1}{2}.(3) + \frac{7}{2} = -\left\{\frac{5}{6}.(-4) - \frac{5}{3}\right\}$ like wise.

Lemma 1: A rational and real quantity n can always be expressed as $n = a.x + A$ where a, x and A are rational quantities. If a is fixed, even then x and A can have infinite rational real values that satisfy above said equation. If a and x are fixed then A also gets fixed at one and only one value.

If rational quantities X, Y, Z and W satisfy relation $X^3 + Y^3 + X^3 = W^3$, then using lemma 1, the relation can be written as $a_1^3.x^3 + (a_2.x + A_2)^3 + (A_3 - a_2.x)^3 = (a_1.x + A_4)^3$where $a_1, a_2, A_2,\ A_3$and A_4 are real rational values assigned by us. For solving this cubic equation, X, Y, Z and W are written using lemma 1, so that terms containing x^3 may cancel each each other. After expansion, $a_1^3.x^3 + (a_2.x + A_2)^3 + (A_3 - a_2.x)^3 = (a_1.x + A_4)^3$is reduced to a quadratic equation written below

$$x^2.(a_2^2.A_2 + a_2^2.A_3 - a_1^2.A_4) + x.(a_2.A_2^2 - a_2.A_3^2 - a_1.A_4^2) = \frac{A_4^3 - A_3^3 - A_2^3}{3} \tag{2}$$

and has roots,

$$x = \left[\frac{a_1.A_4^2 + a_2.A_3^2 - a_2.A_2^2 \pm \sqrt{(a_1A_4^2 + a_2.A_3^2 - a_2.A_2^2)^2 + 4.\frac{(a_2^2.A_2 + a_2^2.A_3 - a_1^2.A_4)(A_4^3 - A_3^3 - A_2^3)}{3}}}{2.(a_2^2.A_2 + a_2^2.A_3 - a_1^2.A_4)}\right] \tag{3}$$

For roots to be real and rational, quantity under square root must be a whole square and that requires careful selection of $a_1, a_2, A_2,\ A_3$and A_4. It is cumbersome to have those particular values of $a_1, a_2, A_2,\ A_3$ and A_4. To obviate this difficulty, coefficient of x^2 is made zero to make it a linear equation. That is $(a_2^2.A_2 + a_2^2.A_3 - a_1^2.A_4) = 0.$

$$\text{Or } A_4 = \frac{a_2^2}{a_1^2}.(A_2 + A_3) \tag{4}$$

That makes

$$x = \frac{A_3^3 + A_2^3 - A_4^3}{3.\{a_1.A_4^2 + a_2.(A_3^2 - A_2^2)\}}. \tag{5}$$

Therefore

$$\left\{\frac{a_1(A_3^3 + A_2^3 - A_4^3)}{3.(a_1.A_4^2 + a_2.A_3^2 - a_2.A_2^2).}\right\}^3 + \left\{\frac{a_2.(A_3^3 + A_2^3 - A_4^3)}{3.(a_1.A_4^2 + a_2.A_3^2 - a_2.A_2^2).} + A_2\right\}^3 + \left\{A_3 - \frac{a_2(A_3^3 + A_2^3 - A_4^3)}{3.(a_1.A_4^2 + a_2.A_3^2 - a_2.A_2^2).}\right\}^3 = \left\{\frac{a_1(A_3^3 + A_2^3 - A_4^3)}{3.(a_1.A_4^2 + a_2.A_3^2 - a_2.A_2^2).} + A_4\right\}^3 \tag{6}$$

2. THEORY AND CONCEPT

Lemma 2: $a_1^3.x^3 + (a_2.x + A_2)^3 + (A_3 - a_2.x)^3$ always equals to $(a_1.x + A_4)^3$ when $A_4 = \frac{a_2^2}{a_1^2}.(A_2 + A_3), x = \frac{A_3^3 + A_2^3 - A_4^3}{3.\{a_1.A_4^2 + a_2.(A_3^2 - A_2^2)\}.}$ and $a_1, a_2, A_2,\ A_3, A_4$ are real and rational quantities.

On assigning different values to $a_1, a_2, A_2,\ A_3$ and A_4 and also A_4, satisfying equation (4), different sets of numbers like those of Taxicab and Cab taxi are generated after normalisation. Here normalisation means multiplying each term with least common multiplier LCM so as to make these integers. Some of these numbers satisfying equation (4) are given in Table 1 and Table 2.

Table 1. Numbers Satisfying Taxicab Relation

S.N.	a_1	a_2	A_2	A_3	A_4	Unnormalised 'x'	Normalised X^3+Y^3	Normalised W^3-Z^3
1	2	1	3	5/2	11/8	207/16	414^3+255^3	436^3+167^3
2	2	1	11/4	5/2	21/16	2221/416	4442^3+3365^3	1181^3+4988^3
3	2	1	3	11/4	23/16	887/160	1774^3+1367^3	447^3+2004^3
4	1	-2	3	-4	-4	9/2	9^3+10^3	12^3+1^3
5	2	1	11/5	9/5	1	129/10	258^3+151^3	111^3+268^3
6	2	1	21/10	19/10	1	21/5	84^3+63^3	23^3+94^3
7	2	1	41/20	39/20	1	501/160	1002^3+829^3	189^3+1162^3
8	2	1	51/25	49/25	1	149/50	298^3+251^3	51^3+348^3
9	2	1	43/20	37/20	1	509/80	1018^3+681^3	361^3+1098^3
10	2	1	17/8	15/8	1	81/16	162^3+115^3	51^3+178^3
11	2	1	33/16	31/16	1	107/32	214^3+173^3	45^3+246^3
12	2	1	81/40	79/40	1	667/240	1334^3+1153^3	193^3+1574^3
13	2	1	101/50	99/50	1	1563/575	6252^3+5449^3	849^3+7402^3
14	2	1	65/32	63/32	1	183/64	366^3+313^3	57^3+430^3
15	2	1	129/64	127/64	1	1707/640	3414^3+2997^3	437^3+4054^3

Table 2. Numbers Satisfying Cab-taxi Relation

S.N.	a_1	a_2	A_2	A_3	A_4	Unnormalise d 'x'	Normalised X^3+Y^3	Normalised W^3-Z^3
1	1	2	3	1	16		113^3+166^3	246^3-207^3
2	1	2	3	-2	4		5^3+4^3	6^3-3^3
3	1	2	3	-4	-4	3/10	3^3+36^3	46^3-37^3
4	1	2	3	-5	-8	23/16	23^3+94^3	126^3-105^3
5	1	2	3	-1	8	-27/8	27^3+30^3	46^3-37^3
6	1	2	3	-1/2	10	-173/44	173^3+214^3	324^3-267^3
7	1	2	3	-6	-12	57/22	57^3+180^3	246^3-207^3
8	1	-2	3	1	16	-339/68	-339^3+882^3	610^3+749^3
9	1	-2	3	-2	4	-15/26	15^3+82^3	108^3-89^3
10	1	-2	3	-5	-8	69/16	69^3+58^3	90^3-59^3
11	1	-2	3	-1	8	-81/40	-81^3+282^3	239^3+202^3
12	1	-2	3	-1/2	10	-519/188	-529^3+1602^3	1132^3+1361^3
13	1	-2	-3	2	-4	15/26	108^3-15^3	89^3+82^3
14	-1	-2	-2	4	8	19/11	19^3+60^3	82^3-69^3
15	1	3	1	-2	-9	361/135	361^3+1218^3	1353^3-854^3

— Algebraic sum of cubes that equals another cube in integer form when number of terms are even

While generating numbers of the like of Taxicab and Cab taxi, only four terms were considered, now formulae for generating sum of cubes that equals another cube when number of terms are more than four will be considered. In the first instance, number of terms say n which is even will be considered. Let us say $X_1^3+X_2^3+X_3^3+X_4^3+X_5^3=X_6^3$ where $X_1, X_2, X_3, \ldots X_6$ all are integers positive or negative. Using lemma 1, let $X_1 = a_1.x + A_1$, $X_2 = a_2.x + A_2$, $X_3 = A_3 - a_2.x$, $X_4 = a_4.x + A_4$, $X_5 = A_5 - a_4.x$, $X_6 = a_1.x + A_6$ where $A_1, A_2, A_3, A_4, A_5, A_6, a_1, a_2, a_4$ are real and rational quantities. Then

$$(a_1.x+A_1)^3+(a_2.x+A_2)^3+(A_3-a_2.x)^3+(a_4.x+A_4)^3+(A_5-a_4.x)^3=(a_1.x+A_6)^3.$$

On expanding,

$$3.x^2(a_1^2.A_1+a_2^2.A_2+a_2^2.A_3+a_4^2.A_4+a_4^2.A_5-a_1^2.A_6)+3.x(a_1.A_1^2+a_2.A_2^2-a_2.A_3^2+a_4.A_4^2-a_4.A_5^2-a_1.A_6^2) = A_6^3-(A_1^3+A_2^3+A_3^3+A_4^3+A_5^3).$$

For making coefficient of x^2 equal to zero,

$$A_6 = A_1 + \frac{\{a_2^2.(A_2+A_3)+a_4^2.(A_4+A_5)\}}{a_1^2} \quad (7)$$

Then

$$x = \frac{A_6^3-(A_1^3+A_2^3+A_3^3+A_4^3+A_5^3)}{3.\{a_1.(A_1^2-A_6^2)+a_2(A_2^2-A_3^2)+a_4.(A_4^2-A_5^2)\}} \quad (8)$$

when n = 8, $(a_1.x+A_1)^3+(a_2.x+A_2)^3+(A_3-a_2.x)^3+(a_4.x+A_4)^3+(A_5-a_4.x)^3+(a_6.x+A_6)^3+(A_7-a_6.x)^3=(a_1.x+A_8)^3$

$$3.x^2\{a_1^2.A_1 + a_2^2.(A_2 + A_3) + a_4^2.(A_4 + A_5) + a_6^2.(A_6 + A_7) - a_1^2.A_8\} + 3.x\{a_1.A_1^2 + a_2.(A_2^2 - A_3^2) + a_4.(A_4^2 - A_5^2) + a_6.(A_6^2 - A_7^2) - a_1.A_8^2\} = A_8^3 - (A_1^3 + A_2^3 + A_3^3 + A_4^3 + A_5^3 + A_6^3 + A_7^3).$$

and $A_8 = A_1 + \frac{\{a_2^2.(A_2+A_3)+a_4^2.(A_4+A_5)\}+a_6^2(A_6+A_7)}{a_1^2}$, $x = \frac{A_8^3-(A_1^3+A_2^3+A_3^3+A_4^3+A_5^3+A_6^3+A_7^3)}{3.\{a_1.(A_1^2-A_8^2)+a_2(A_2^2-A_3^2)+a_4.(A_4^2-A_5^2)+a_6.(A_6^2-A_7^2)\}}$

Generalising it for even n, $(a_1.x + A_1)^3 + (a_2.x + A_2)^3 + (A_3 - a_2.x)^3 + (a_4.x + A_4)^3 + (A_5 - a_4.x)^3 + \cdots + (A_{n-1} - a_{n-2}.x)^3 = (a_1.x + A_n)^3$ then

$$A_n = A_1 + \frac{\{a_2^2.(A_2+A_3)+a_4^2.(A_4+A_5)\}+a_6^2(A_6+A_7)+\cdots a_{n-2}^2.(A_{n-2}+A_{n-1})}{a_1^2}, \quad (9)$$

and

$$x = \frac{A_n^3-(A_1^3+A_2^3+A_3^3+\cdots+A_{n-1}^3)}{3.\{a_1.(A_1^2-A_n^2)+a_2(A_2^2-A_3^2)+\cdots+a_{n-2}.(A_{n-2}^2-A_{n-1}^2)\}} \quad (10)$$

$$\left[a_1.\frac{A_n^3 - (A_1^3 + A_2^3 + A_3^3 + \cdots + A_{n-1}^3)}{3.\{a_1.(A_1^2 - A_n^2) + a_2(IA_2^2 - A_3^2) + \cdots + a_{n-2}.(A_{n-2}^2 - A_{n-1}^2)\}} + A_1\right]^3 + \left[a_2.\frac{A_n^3 - (A_1^3 + A_2^3 + A_3^3 + \cdots + A_{n-1}^3)}{3.\{a_1.(A_1^2 - A_n^2) + a_2(A_2^2 - A_3^2) + \cdots + a_{n-2}.(A_{n-2}^2 - A_{n-1}^2)\}} + A_2\right]^3 + \left[A_3 - a_2.\frac{A_n^3 - (A_1^3 + A_2^3 + A_3^3 + \cdots + A_{n-1}^3)}{3.\{a_1.(A_1^2 - A_n^2) + a_2(A_2^2 - A_3^2) + \cdots + a_{n-2}.(A_{n-2}^2 - A_{n-1}^2)\}}\right]^3 + \cdots + \left[A_{n-1} - a_{n-2}.\frac{A_n^3 - (A_1^3 + A_2^3 + A_3^3 + \cdots + A_{n-1}^3)}{3.\{a_1.(A_1^2 - A_n^2) + a_2(A_2^2 - A_3^2) + \cdots + a_{n-2}.(A_{n-2}^2 - A_{n-1}^2)\}}\right]^3 = \left[a_1.\frac{A_n^3-(A_1^3+A_2^3+A_3^3+\cdots+A_{n-1}^3)}{3.\{a_1.(A_1^2-A_n^2)+a_2(A_2^2-A_3^2)+\cdots+a_{n-2}.(A_{n-2}^2-A_{n-1}^2)\}} + A_n\right]^3 \quad (11)$$

On the basis of the above said formula, for some even values of n, equations of sum of cubes those equal an another cube are given in the Table 3.

Table 3. Sum of cubes that equals an another cube

Value of n or number of terms	Values of a's And A's	Unnormalised Value Of x	Normalised Sum Of Cubes $X_1^3 + X_2^3 + X_3^3 + \cdots + X_{n-1}^3 = X_n^3$
6	$A_1 = 1, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = 14, a_1 = 1, a_2 = 2, a_4 = 3$	$-\frac{2717}{648}$	$-2069^3 - 6082^3 + 6730^3 - 9497^3 + 10095^3 = 6535^3$
8	$A_1 = 1, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = \frac{15}{8}, a_1 = 4, a_2 = 3, a_4 = 2, a_6 = 1.$	$\frac{29393}{55392}$	$172964^3 + 32787^3 + 22605^3 - 51998^3 + 107390^3 - 136783^3 + 192175^3 = 221432^3$
10	$A_1 = 1, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = \frac{11}{5}, = 5, a_2 = 4, a_4 = 3, a_6$	$\frac{7147}{12975}$	$48710^3 + 15613^3 - 2638^3 - 4509^3 + 17484^3 - 24631^3 + 37606^3 - 44753^3 + 57728^3 = 64280^3$
12	$A_1 = 1, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = -5, A_{11} = 6, A_{12} = \frac{7}{2}, a_1 = -6, a_2 = 2, a_4 = 3, a_6 = 4, a_8 = 5, a_{10} = 6$	$\frac{277}{444}$	$-1218^3 + 110^3 + 334^3 - 57^3 + 501^3 - 224^3 + 668^3 - 391^3 + 835^3 - 558^3 + 1002^3 = -108^3$
14	$A_1 = 0, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = -5, A_{11} = 6, A_{12} = -6, A_{13} = 5, A_{14} = 1, = -3, a_2 = 2, a_4 = 3, a_6 = 4, a_8 = 5, a_{10} = 6, a_{12} = 9.$	$\frac{41}{58}$	$-123^3 + 24^3 + 34^3 + 7^3 + 51^3 - 10^3 + 68^3 - 27^3 + 85^3 - 44^3 + 102^3 + 21^3 - 79^3 = -65^3$

16	$A_1 = 0, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = -5, A_{11} = 6, A_{12} = -6, A_{13} = 7, A_{14} = -7,\ A_{15} = 6, A_{16} = 2, = -3, a_2 = 2, a_4 = 3, a_6 = 4, a_8 = 5, a_{10} = 6, a_{12} = 7, a_{14} = 11.$	$\frac{23}{32}$	$-69^3 + 14^3 + 18^3 + 5^3 + 27^3 - 4^3 + 36^3 - 13^3 + 45^3 - 22^3 + 54^3 - 31^3 + 63^3 + 29^3 - 61^3 = -5^3$
18	$A_1 = 0, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = -5, A_{11} = 6, A_{12} = -6, A_{13} = 7, A_{14} = -7,\ A_{15} = 8, A_{16} = -8, A_{17} = 7, A_{18} = 7,$ $= 1, a_2 = 2, a_4 = 3, a_6 = 4, a_8 = 5, a_{10} = 6, a_{12} = 7, a_{14} = 8, a_{16} = 14.$	$-\frac{1}{630}$	$-1^3 - 632^3 + 1262^3 - 1263^3 + 1893^3 - 1894^3 + 2524^3 - 2525^3 + 3155^3 - 3156^3 + 3786^3 - 3787^3 + 4417^3 - 4418^3 + 5048^3 - 5054^3 + 4424^3 = 4409^3$
20	$A_1 = 0, A_2 = -1, A_3 = 2, A_4 = -2, A_5 = 3, A_6 = -3, A_7 = 4, A_8 = -4, A_9 = 5, A_{10} = -5, A_{11} = 6, A_{12} = -6, A_{13} = 7, A_{14} = -7,\ A_{15} = 8, A_{16} = -8, A_{17} = 9, A_{18} = -9,\ A_{19} = 8, A_{20} = 7,\ a_1 = -2, a_2 = 2, a_4 = 3, a_6 = 4, a_8 = 5, a_{10} = 6, a_{12} = 7, a_{14} = 8, a_{16} = 9, a_{16} = 16.$	$\frac{4}{11}$	$-8^3 - 3^3 + 14^3 - 10^3 + 21^3 - 17^3 + 28^3 - 24^3 + 35^3 - 31^3 + 42^3 - 38^3 + 49^3 - 45^3 + 56^3 - 52^3 + 63^3 - 35^3 + 24^3 = 69^3$

— Algebraic sum of cubes that equals another cube in integer form when number of terms are even or odd

It is not always the case that 'n' is even, it may be odd also. To take care of that a method has been devised whereby an empirical equation is found and that equation is then utilised to further find sum of cubes that equals another cube.

Lemma 3: If m is a positive integer such that $n-1$ lies between $(m-1)^3$ and m^3 then $(m+p)^3 - (n-1)$ can always equal $x_2.(7) + x_3.(26) + x_4.(63) + \cdots + x_k.(k^3-1)$ where $x_2, x_3, \ldots x_k$ and p are positive integers (including zero) such that $(x_2 + x_3 + x_4 + \cdots + x_k) \le (n-1)$.

Lemma 4: If m is an integer such that $n-1$ lies between $(m-1)^3$ and m^3 and $(m+p)^3 - (n-1) = x_2.(7) + x_3.(26) + x_4.(63) + \cdots + x_k.(k^3-1)$ then $(m+p)^2 = x_1.(1^2) + x_2.(2^2) + x_3.(3^2) + x_4.(4^2) + \cdots + x_k.(k^2)$ where $x_1\ x_2, x_3, \ldots x_k$ and p are positive integers (including zero) such that $(x_2 + x_3 + x_4 + \cdots + x_k) \le (n-1)$ and $(x_1 + x_2 + x_3 + \cdots + x_k) = (n-1)$

Proof: It is obvious from lemma 3 and 4 that sum of cubes may have terms that repeat but final equation generated from it, will have terms differing from one another unless our requirement demands otherwise. Let there be all unit cubes, then

$$1^3 + 1^3 + 1^3 \ldots \text{upto } n-1 \text{ terms} = n-1 \quad (12)$$

We will find positive integer 'm' so that $(n-1)$ lies between $(m-1)^3$ and m^3. That is $(n-1) > (m-1)^3$ and $(n-1) < m^3$. Value of m can thus always be determined and with that determination, value of $m^3 - (n-1)$ which is always positive integer since m and n are both positive and integers, can always be found. For generalised form, we take $(m+p)^3 - (n-1)$ where p is any positive integer 0, 1, 2, 3…..so on. Obviously, $(m+p)^3 - (n-1)$ will again be positive integer. Putting $(m+p)^3 - (n-1)$ in identity (12), we get

$$(m+p)^3 - (1^3 + 1^3 + 1^3 .. \text{upto } n-1 \text{ terms}) = (m+p)^3 - (n-1) \quad (13)$$

$$\text{Let } (m+p)^3 - (n-1) = x_2.(7) + x_3.(26) + x_4.(63) + \cdots x_k.(k^3-1) \quad (14)$$

where $x_2, x_3, x_4, \ldots x_k$ are positive integers including 0 such that $(x_2 + x_3 + x_4 + \cdots + x_k) \le (n-1)$. Equation (14) is always achievable by increasing value of p from 0 to 1 to 2 to 3 so on till it is satisfied. That proves lemma 3. On putting the value of $(m+p)^3 - (n-1)$ from equation (14) into equation (13),

$$(m+p)^3 - (1^3 + 1^3 + 1^3 .. \text{upto } n-1 \text{ terms}) = x_2.(7) + x_3.(26) + x_4.(63) + \cdots x_k.(k^3-1).$$

On rearranging ,

$$(m+p)^2 = (x_1).(1^3) + x_2.(2^3) + (x_3).(3^3) + (x_4).(4^3) + \cdots (x_k).(k^3) \quad (15)$$

where $x_1 = \{(n-1) - (x_2 + x_3 + x_4 + \cdots + x_k)\}$ and can have any integer value including 0. That makes

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_k = n - 1. \quad (15/1)$$

Derivations of equations (15) and (15/1) prove lemma 4. On splitting up, equation (15) can be written as

$$(m+p)^3 = [1^3 + 1^3 + 1^3 + \cdots \text{repeated } x_1 \text{ times}] + (2^3 + 2^3 + 2^3 + \cdots \text{repeated } x_2 \text{ times}) + (3^3 + 3^3 + 3^3 + \cdots \text{repeated } x_3 \text{ times}) + \cdots + (k^3 + k^3 + k^3 + \cdots \text{repeated } x_k \text{ times}) \quad (16)$$

Left hand side of equation (16) has a perfect cube $(m+p)^3$ and right hand side has sum of cubes of (n – 1) terms, therefore, total number of terms are n and it fulfils all essential ingredients to be sum of cubes equalling another cube when number of terms is n. Given below in Table 4 are some of such seed equations where n varies from 4 to 10^3. We have given a single seed equation for different n although there can be infinite number of such equations as the value of p is increased.

Table 4. Seed Equations For Different Value Of Integers 'n'.

n	$m+p$	$(m+p)^3 = x_1.(1^3) + x_2.(2^3) + x_3.(3^3) + \cdots x_k.(k^3)$
3		According to Fermat's last theorem, a^3 can never be equal to $b^3 + c^3$ for integer value of a, b and c.
4	$2+4$	$6^3 = 1.(3^3) + 1.(4^3) + 1.(5^3)$
5	$2+5$	$7^3 = 2.(1^3) + 1.(5^3) + 1.(6^3)$
6	$2+2$	$4^3 = 2.(1^3) + 1.(2^3) + 2.(3^3)$
7	$2+1$	$3^3 = 3.(1^3) + 3.(2^3)$
8	$2+4$	$6^3 = 2.(1^3) + 1.(2^3) + 3.(3^3) + 1.(5^3)$
10	$3+5$	$8^3 = 2.(1^3) + 3.(2^3) + 2.(3^3) + 2.(6^3)$
11	$3+1$	$4^3 = 5.(1^3) + 4.(2^3) + 1.(3^3)$
12	$3+2$	$5^3 = 4.(1^3) + 5.(2^3) + 1.(3^3) + 1.(4^3)$
13	$3+1$	$4^3 = 10.(1^3) + 2.(3^3)$
14	$3+0$	$3^3 = 11.(1^3) + 2.(2^3)$
15	$3+2$	$5^3 = 9.(1^3) + 1.(2^3) + 4.(3^3)$
20	$3+2$	$5^3 = 12.(1^3) + 4.(2^3) + 3.(3^3)$
30	$4+0$	$4^3 = 24.(1^3) + 5.(2^3)$
50	$4+2$	$6^3 = 44.(1^3) + 4.(3^3) + 1.(4^3)$
100	$5+0$	$5^3 = 98.(1^3) + 1.(3^3)$
500	$8+1$	$9^3 = 493.(1^3) + 4.(3^3) + 2.(4^3)$
1000	$10+1$	$11^3 = 989.(1^3) + 4.(2^3) + 2.(3^3) + 4.(4^3)$

— A populating equation of sum of cubes equalling another cube from seed equation

Lemma 6: When integer $n > 3$, m is a positive integer such that $n-1$ lies between $(m-1)^3$ and m^3, p, $x_1, x_2, x_3 \ldots x_k$ are positive integers, $\{(a_{11}, a_{12}, a_{13}, \ldots a_{1x_1}), (a_{21}, a_{22}, a_{23}, \ldots a_{2x_2}), (a_{31}, a_{32}, a_{33}, \ldots a_{3x_3}) \ldots (a_{k1}, a_{k2}, a_{k3}, \ldots a_{kx_k})\}$ are real rational quantities assigned and $(m+p)^3 = x_1.(1^3) + x_2.(2^3) + x_3.(3^3) + x_4.(4^3) + \cdots + x_k.(k^3)$ then

$$\{(x+a_{11})^3 + (x+a_{12})^3 + (x+a_{13}) + \ldots \text{upto } (x+a_{1x_1})\} + \{(2.x+a_{21})^3 + (2.x+a_{22})^3 + (2.x+a_{23}) + \cdots \text{upto } (2.x+a_{2x_2})^3\} + \{(3.x+a_{31})^3 + (3.x+a_{32})^3 + (3.x+a_{33})^3 + \ldots \text{upto } (3.x+a_{3x_3})^3\} + \{(k.x+a_{k1})^3 + (k.x+a_{k2})^3 + (k.x+a_{k3})^3 + \ldots \text{upto } (k.x+a_{kx_k})^3\} = \{(m+p).x+a\}^3$$

and after normalisation gives integer values of cubes provided 'a' has value equal to

$$\frac{\{(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1})+2^2.(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2})+3^2.(a_{31}+a_{32}+a_{33}+\cdots+a_{3x_3})+\cdots+k^2.(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k})\}}{(m+p)^2}$$

and x has value equal to

$$\frac{[a^3-\{(a_{11}^3+a_{12}^3+a_{13}^3+\cdots+a_{1x_1}^3)+(a_{21}^3+a_{22}^3+a_{23}^3+\cdots+a_{2x_2}^3)+(a_{31}^3+a_{32}^3+a_{33}^3+\cdots+a_{3x_3}^3)+\cdots+(a_{k1}^3+a_{k2}^3+a_{k3}^3+\cdots a_{kx_k}^3)\}]}{3.\{(a_{11}^2+a_{12}^2+a_{13}^2+\cdots a_{1x_1}^2)+2.(a_{21}^2+a_{22}^2+a_{23}^2+\cdots a_{2x_2}^2)+3.(a_{31}^2+a_{32}^2+a_{33}^2+\cdots a_{3x_3}^2)+\cdots+k.(a_{k1}^2+a_{k2}^2+a_{k3}^2+\cdots a_{kx_k}^2)-(m+p).a^2\}}$$

Proof: To utilise seed equation of sum of cubes that equals another cube given by equation (16), using lemma 1, we will write the terms

$x_1(1^3)$ i.e $(1^3 + 1^3 + 1^3 + \cdots \text{repeated } x_1 \text{times})$ as

$$\{(x+a_{11})^3 + (x+a_{12})^3 + (x+a_{13})^3 + \cdots + (x+a_{1x_1})^3\},$$

$x_2.(2^3)$ i.e. $(2^3 + 2^3 + 2^3 + \cdots \text{repeated } x_2 \text{ times})$ as

$$\{(2.x+a_{21})^3 + (2.x+a_{22})^3 + (2.x+a_{23})^3 + \cdots + (2.x+a_{2x_2})^3\},$$

$x_3.(3^3)$ i.e. $(3^3 + 3^3 + 3^3 + \cdots \text{repeated } x_3 \text{ times})$ as

$$\left\{(3.x + a_{31})^3 + (3.x + a_{32})^3 + (3.x + a_{33})^3 + \cdots + \left(3.x + a_{3x_3}\right)^3\right\},$$

$x_k.(k^3)$ i.e. $(k^3 + k^3 + k^3 + \cdots \text{repeated } x_k \text{times})$ as

$$\left\{(k.x + a_{k1})^3 + (k.x + a_{k2})^3 + (k.x + a_{k3})^3 + \cdots + \left(k.x + a_{kx_k}\right)^3\right\},$$

and term $(m + p)^3$ as single term $\{(m + p).x + a\}^3$. Then

$$\left\{(x + a_{11})^3 + (x + a_{12})^3 + (x + a_{13})^3 + \cdots + \left(x + a_{1x_1}\right)^3\right\} + \left\{(2.x + a_{21})^3 + (2.x + a_{22})^3 + (2.x + a_{23})^3 + \cdots + \left(2.x + a_{2x_2}\right)^3\right\} + \left\{(3.x + a_{31})^3 + (3.x + a_{32})^3 + (3.x + a_{33})^3 + \cdots + \left(3.x + a_{3x_2}\right)^3\right\} + \cdots + \left\{(k.x + a_{k1})^3 + (k.x + a_{k2})^3 + (k.x + a_{k3})^3 + \cdots + \left(k.x + a_{kx_k}\right)^3\right\} = \{(m + p).x + a\}^3 \quad (17)$$

On expanding and arranging, coefficients of x^3 will cancel as $(m + p)^3 = x_1.(1^3) + x_2.(2^3) + x_3.(3^3) + x_4.(4^3) + \cdots + x_k.(k^3)$ and we will be left with equation,

$$3.x^2.\left\{\left(a_{11} + a_{12} + a_{13} + \cdots + a_{1x_1}\right) + 2^2.\left(a_{21} + a_{22} + a_{23} + \cdots + a_{2x_2}\right) + 3^2.\left(a_{31} + a_{32} + a_{33} + \cdots + a_{3x_3}\right) + \cdots + k^2.\left(a_{k1} + a_{k2} + a_{k3} + \cdots + a_{kx_k}\right)\right\} + 3.x.\left\{\left(a_{11}^2 + a_{12}^2 + a_{13}^2 + \cdots + a_{1x_1}^2\right) + 2.\left(a_{21}^2 + a_{22}^2 + a_{23}^2 + \cdots + a_{2x_2}^2\right) + 3.\left(a_{31}^2 + a_{32}^2 + a_{33}^2 + \cdots + a_{3x_3}^2\right) + \cdots + k.\left(a_{k1}^2 + a_{k2}^2 + a_{k3}^2 + \cdots a_{kx_k}^2\right)\right\} + \left\{\left(a_{11}^3 + a_{12}^3 + a_{13}^3 + \cdots + a_{1x_1}^3\right) + \left(a_{21}^3 + a_{22}^3 + a_{23}^3 + \cdots + a_{2x_2}^3\right) + \left(a_{31}^3 + a_{32}^3 + a_{33}^3 + \cdots + a_{3x_3}^3\right) + \cdots + \left(a_{k1}^3 + a_{k2}^3 + a_{k3}^3 + \cdots a_{kx_k}^3\right)\right\} = 3.x^2.(m + p)^2.a + 3.x.(m + p).a^2 + a^3$$

This is a quadratic equation in x and has two roots. To obviate the possibility of irrational roots of this quadratic equation, it is reduced to linear equation by equating coefficient of x^2 to zero. That is

$$\left\{\left(a_{11} + a_{12} + a_{13} + \cdots + a_{1x_1}\right) + 2^2.\left(a_{21} + a_{22} + a_{23} + \cdots + a_{2x_2}\right) + 3^2.\left(a_{31} + a_{32} + a_{33} + \cdots + a_{3x_3}\right) + \cdots + k^2.\left(a_{k1} + a_{k2} + a_{k3} + \cdots + a_{kx_k}\right)\right\} = (m + p)^2.a$$

Or '**a**' on right hand side must be equal to

$$\frac{\left\{\left(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1}\right)+2^2.\left(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2}\right)+3^2.\left(a_{31}+a_{32}+a_{33}+\cdots+a_{3x_3}\right)+\cdots+k^2.\left(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k}\right)\right\}}{(m+p)^2} \quad (18)$$

Then **x** equals

$$\frac{\left[a^3-\left\{\left(a_{11}^3+a_{12}^3+a_{13}^3+\cdots+a_{1x_1}^3\right)+\left(a_{21}^3+a_{22}^3+a_{23}^3+\cdots+a_{2x_2}^3\right)+\left(a_{31}^3+a_{32}^3+a_{33}^3+\cdots+a_{3x_3}^3\right)+\cdots+\left(a_{k1}^3+a_{k2}^3+a_{k3}^3+\cdots a_{kx_k}^3\right)\right\}\right]}{3.\left\{\left(a_{11}^2+a_{12}^2+a_{13}^2+\cdots a_{1x_1}^2\right)+2.\left(a_{21}^2+a_{22}^2+a_{23}^2+\cdots a_{2x_2}^2\right)+3.\left(a_{31}^2+a_{32}^2+a_{33}^2+\cdots a_{3x_3}^2\right)+\cdots+k.\left(a_{k1}^2+a_{k2}^2+a_{k3}^2+\cdots a_{kx_k}^2\right)-(m+p).a^2\right\}} \quad (19)$$

where $\left\{\left(a_{11}, a_{12}, a_{13}, \ldots a_{1x_1}\right), \left(a_{21}, a_{22}, a_{23}, \ldots a_{2x_2}\right), \left(a_{31}, a_{32}, a_{33}, \ldots a_{3x_3}\right) \ldots \left(a_{k1}, a_{k2}, a_{k3}, \ldots a_{kx_k}\right)\right\}$, are real rational quantities assigned by us, values of m and p are known from seed equation and value of '**a**' is found by using equation (18). When x is determined according to equation (19),values of cubes are given after normalisation of equation (17) by multiplying with LCM. It is stated that a_{kx_k} denotes x_kth term of k^3, that is k^3 is repeatedly added x_k times and a_{kx_k} is its x_kth term.

To illustrate the formulae (19) so derived, an example is given for finding values of cubes when n = 4. That means $n - 1 = 3$ and $3^{\frac{1}{3}}$ must lie between $(m - 1)$ and m . Or positive integer m must lie between $\left(3^{\frac{1}{3}} + 1\right)$ and $3^{\frac{1}{3}}$. Since 2 is the integer that lies between $\left(3^{\frac{1}{3}} + 1\right)$ and $3^{\frac{1}{3}}$, therefore, $m = 2$. Now $(m + p)^3 - (n - 1)$ when p = 0 i.e $2^3 - 3 = 5$, is not expressible as $x_2.(7) + x_3.(26) + x_4.(63) + \cdots x_k.(k^3 - 1)$, therefore trying for p = 2, or 3 so on, it is found that at p = 4, $(m + p)^3 - (n - 1) = 6^3 - 3 = 213 = 1.(26) + 1.(63) + 1.(125)$. Or $6^3 = 3^3 + 4^3 + 5^3$.

Let $a_{31} = 1, a_{41} = -1, a_{51} = 1$ then $a = \frac{3^2-4^2+5^2}{6^2} = \frac{1}{2}$, $x = \frac{\left\{\left(\frac{1}{2}\right)^3-(1-1+1)\right\}}{3\left\{3.1^2+4.(-1)^2+5.1^2-6.\left(\frac{1}{2}\right)^2\right\}} = -\frac{1}{36}$ and

$(3.x + a_{31})^3 + (4.x + a_{41})^3 + (5.x + a_{51})^3 = (6.x + a)^3$.

On putting the values of $a_{31} = 1, a_{41} = -1, a_{51} = -1, a = \frac{1}{2}, x = -\frac{1}{36}$, and after normalising, $33^3 - 40^3 + 31^3 = 12^3$.

Based on seed equation, $6^3 = 1.(3^3) + 1.(4^3) + 1.(5^3)$, a number of equations, from

$$(3.x + a_{31})^3 + (4.x + a_{41})^3 + (5.x + a_{51})^3 = (6.x + a)^3$$

can be obtained by assigning values to a_{31}, a_{41}, a_{51} in such a way that $a = \frac{3^2.a_{31}+4^2.a_{41}+5^2.a_{51}}{6^2}$ where

$$x = \frac{a^3 - a_{31}^3 - a_{41}^3 - a_{51}^3}{3.(3.a_{31}^2 + 4.a_{41}^2 + 5.a_{51}^2 - 6.a^2)}$$

Equations those are of the form $X^3 + Y^3 = W^3 + Z^3$ that is like Taxicab Numbers are given in Table 5 and those of the form $X^3 + Y^3 = W^3 - Z^3$ that is like Cab Taxi Number are given in Table 6.

Table 5. Sum of cubes of the form Taxicab Number i.e. $X^3 + Y^3 = W^3 + Z^3$

S.N	a_{31}	a_{41}	a_{51}	Calculated 'a'	Calculated 'x'	Normalised $(3.x + a_{31})^3 + (4.x + a_{41})^3 + (5.x + a_{51})^3 = (6.x + a)^3$
1	1	-1	1	1/2	-1/36	$33^3 - 40^3 + 31^3 = 12^3$
2	1	-1	-1	-8/9	31/2268	$2361^3 - 2144^3 - 2113^3 = -1830^3$
3	-1	-1	-1	-25/18	1871/7452	$-1842^3 + 32^3 + 1903^3 = 876^3$
4	1	-2	2	3/4	-37/6840	$6728^3 - 13828^3 + 13495^3 = 4908^3$
5	2	1	-1	1/4	-511/3960	$6387^3 + 1916^3 - 6515^3 = 2076^3$
6	-3	1	-1	-1	13/45	$-96^3 + 97^3 + 20^3 = 33^3$
7	3	1	-2	1/2	-225/828	$1839^3 - 32^3 - 1903 \quad = -876^3$
8	-3	2	-2	-5/4	1603/109296	$-26079^3 + 27004^3 - 12577^3 = -3252^3$
9	-4	1	-4	-10/3	347/756	$-1983^3 + 2144^3 - 1289^3 = -438^3$
10	-4	-1	4	4/3	1/108	$-429^3 - 104^3 + 437^3 = 150^3$
11	-5	-1	1	-1	62/117	$-399^3 + 131^3 + 427^3 = 255^3$
12	5	-3	3	2	-13/44	$181^3 - 184^3 + 67^3 = 10^3$
13	-7	1	-1	-2	335/396	$-1767^3 + 1736^3 + 1279^3 = 1218^3$
14	-7	-1	1	-3/2	143/180	$-831^3 + 392^3 + 895^3 = 588^3$

Table 6. Sum of cubes of the form Cab taxi Number i.e. $X^3 + Y^3 = W^3 - Z^3$

S.No	a_{31}	a_{41}	a_{51}	Calculated 'a'	Calculated 'x'	Normalised $(3.x + a_{31})^3 + (4.x + a_{41})^3 + (5.x + a_{51})^3 = (6.x + a)^3$
1	1	1	-1	0	-1/36	$33^3 + 32^3 - 41^3 = -6^3$
2	-1	-2	2	1/4	65/7416	$-7221^3 - 14572^3 + 15157^3 = 2244^3$
3	2	-1	1	3/4	-485/3384	$5313^3 - 5324^3 + 959^3 = -372^3$
4	3	2	-2	1/4	-1727/12024	$30891^3 + 17140^3 - 32683^3 = -7356^3$
5	-4	-1	4	4/3	1/108	$-429^3 - 104^3 + 437^3 = 150^3$
6	-5	-3	3	-1/2	111/412	$-1727^3 - 792^3 + 1791^3 = 460^3$
7	4	9	0	5	-334/333	$330^3 + 1661^3 - 1670^3 = -339^3$
8	16	9	0	8	-4313/2124	$21045^3 + 1864^3 - 21565^3 = -8886^3$
9	12	9	0	7	-151/99	$735^3 + 287^3 - 755^3 = -213^3$
10	6	9	0	11/2	-6229/6012	$17385^3 + 29192^3 - 31145^3 = -4308^3$

In the same way, using corresponding seed equation from Table 4 and also equations (18) and (19), values of 'a' and 'x' can be found out. Putting values of 'a' and 'x' so obtained in equation (17) and thereafter, multiplying with LCM, sum of cubes of n terms that equals another cube in integer form is obtained. Sums of such cubes for different values of n are given in Table 7.

Table 7. Sum of cubes that equals another cube for varying 'n.'

S. No. of seed equation as given in Table4	Values Of a_{ij}	'a'	'x'	Normalised $X_1^3 + X_2^3 + X_3^3 + \cdots X_{n-1}^3 = X_n^3$
4	$a_{31} = 8, a_{41} = 1, a_{51} = -4.$	-1/3	-866/1593	$10146^3 - 1871^3 - 10702^3 = -5727^3$
5	$a_{11} = 1, a_{12} = -1, a_{51} = -2, a_{61} = 1.$	-2/7	2393/28224	$30617^3 - 25831^3 - 44483^3 + 42582^3 = 8687^3$
6	$a_{11} = 1, a_{12} = -1, a_{21} = 4, a_{31} = 1, a_{32} = -1.$	1	-7/12	$5^3 - 19^3 + 34^3 - 9^3 - 33^3 = -16^3$
7	$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{21} = -2, a_{22} = -3, a_{23} = 0.$	-2	19/60	$79^3 - 41^3 + 139^3 - 82^3 - 142^3 + 38^3 = -63^3$
8	$a_{11} = 2, a_{12} = -1, a_{21} = 2, a_{31} = 3, a_{32} = -2, a_{33} = 0, a_{51} = 0.$	1/2	-271/1212	$2153^3 - 1483^3 + 1882^3 + 2823^3 - 3237^3 - 813^3 - 1355^3 = -1020^3$
10	$a_{11} = 1, a_{12} = -1, a_{21} = 4, a_{22} = 5, a_{23} = -1, a_{31} = 4, a_{32} = -4, a_{61} = 1, a_{62} = -1$	1/2	-167/512	$345^3 - 679^3 + 1714^3 + 2226^3 - 846^3 + 1547^3 - 2549^3 - 490^3 - 1514^3 = -1080^3$

15	$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{14} = -2, a_{15} = 3, a_{16} = -3, a_{17} = 4, a_{18} = -4, a_{19} = 0, a_{21} = 0, a_{31} = 1, a_{32} = 2, a_{33} = -3, a_{34} = 0.$	0	1/17	$18^3 - 16^3 + 35^3 - 33^3 + 52^3 - 50^3 + 69^3 - 67^3 + 1^3 + 2^3 + 20^3 + 37^3 - 48^3 + 3^3 = 5^3$
20	$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{14} = -2, a_{15} = 3, a_{16} = -3, a_{17} = 4, a_{18} = -4, a_{19} = 5, a_{110} = -5, a_{111} = 6, a_{112} = -6, a_{21} = 1, a_{22} = -1, a_{23} = 2, a_{24} = -2, a_{31} = 1, a_{32} = 2, a_{33} = -3,$	0	3/122	$125^3 - 119^3 + 247^3 - 241^3 + 369^3 - 363^3 + 491^3 - 485^3 + 613^3 - 607^3 + 735^3 - 729^3 + 128^3 - 116^3 + 250^3 - 238^3 + 131^3 + 253^3 - 357^3 = 15^3$

To demonstrate veracity of formula (17), (18) and (19) for higher n, some sum of cubes that equals another cube are given in Table 8. Again it is stated, a seed equation corresponding to the value of 'n' is chosen from Table 4 , values are assigned to relevant

$$\{(a_{11}, a_{12}, a_{13}, \dots a_{1x_1}), (a_{21}, a_{22}, a_{23}, \dots a_{2x_2}), (a_{31}, a_{32}, a_{33}, \dots a_{3x_3}) \dots (a_{k1}, a_{k2}, a_{k3}, \dots a_{kx_k})\},$$

value of 'a' is computed from equation (18), value of x from equation (19). Putting these values in equation (17) and multiplying these with LCM give the sum of cubes equal to another cube in integer form.

Table 8. Sum of cubes that equals another cube when 'n' equals 50 and 100

n	Number Of Terms = 50
	$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{14} = -2, a_{15} = 3, a_{16} = -3, a_{17} = 4, a_{18} = -4, a_{19} = 5, a_{110} = -5, a_{111} = 6, a_{112} = -6, a_{113} = 7, a_{114} = -7, a_{115} = 8, a_{116} = -8, a_{117} = 9, a_{118} = -9, a_{119} = 10, a_{120} = -10, a_{121} = 11, a_{122} = -11, a_{123} = 12, a_{124} = -12, a_{125} = 13, a_{126} = -13, a_{127} = 14, a_{128} = -14, a_{129} = 15, a_{130} = -15, a_{131} = 16, a_{132} = -16, a_{133} = 17, a_{134} = -17, a_{135} = 18, a_{136} = -18, a_{137} = 19, a_{138} = -19, a_{139} = 20, a_{140} = -20, a_{141} = 21, a_{142} = -21, a_{143} = 22, a_{144} = -22, a_{31} = 1, a_{32} = 2, a_{33} = 3, a_{34} = -6, a_{41} = 0.$
a	0
x	1/129
Normalised $X_1^3 + X_2^3 + X_3^3 + \dots X_{n-1}^3 = X_n^3$	$130^3 - 128^3 + 259^3 - 257^3 + 388^3 - 386^3 + 517^3 - 515^3 + 646^3 - 644^3 + 775^3 - 773^3 + 904^3 - 902^3 + 1033^3 - 1031^3 + 1162^3 - 1160^3 + 1291^3 - 1289^3 + 1420^3 - 1418^3 + 1549^3 - 1547^3 + 1678^3 - 1676^3 + 1807^3 - 1805^3 + 1936^3 - 1934^3 + 2065^3 - 2063^3 + 2194^3 - 2192^3 + 2323^3 - 2321^3 - 2452^3 - 2450^3 + 2581^3 - 2579^3 + 2710^3 - 2708^3 + 2839^3 - 2837^3 + 132^3 + 261^3 + 390^3 - 771^3 + 4^3 = 6^3$
n	Number Of Terms = 100
a	$a_{11} = 1, a_{12} = -1, a_{13} = 2, a_{14} = -2, a_{15} = 3, a_{16} = -3, a_{17} = 4, a_{18} = -4, a_{19} = 5, a_{110} = -5, a_{111} = 6, a_{112} = -6, a_{113} = 7, a_{114} = -7, a_{115} = 8, a_{116} = -8, a_{117} = 9, a_{118} = -9, a_{119} = 10, a_{120} = -10, a_{121} = 11, a_{122} = -11, a_{123} = 12, a_{124} = -12, a_{125} = 13, a_{126} = -13, a_{127} = 14, a_{128} = -14, a_{129} = 15, a_{130} = -15, a_{131} = 16, a_{132} = -16, a_{133} = 17, a_{134} = -17, a_{135} = 18, a_{136} = -18, a_{137} = 19, a_{138} = -19, a_{139} = 20, a_{140} = -20, a_{141} = 21, a_{142} = -21, a_{143} = 22, a_{144} = -22, a_{145} = 23, a_{146} = -23, a_{147} = 24, a_{148} = -24, a_{149} = 25, a_{150} = -25, a_{151} = 26, a_{152} = -26, a_{153} = 27, a_{154} = -27, a_{155} = 28, a_{156} = -28, a_{157} = 0, a_{158} = 50, a_{159} = 30, a_{160} = -30, a_{161} = 31, a_{162} = -31, a_{163} = 32, a_{164} = -32, a_{165} = 33, a_{166} = -33, a_{167} = 34, a_{168} = -34, a_{169} = 35, a_{170} = -35, a_{171} = 36, a_{172} = -36, a_{173} = 37, a_{174} = -37, a_{175} = 38, a_{176} = -38, a_{177} = 39, a_{178} = -39, a_{179} = 40, a_{180} = -40, a_{181} = 0, a_{182} = 50, a_{183} = 42, a_{184} = -42, a_{185} = 43, a_{186} = -43, a_{187} = 44, a_{188} = -44, a_{189} = 45, a_{190} = -45, a_{191} = 46, a_{192} = -46, a_{193} = 47, a_{194} = -47, a_{195} = 48, a_{196} = -48, a_{197} = 49, a_{198} = -49, a_{31} = 0.$
a	2
x	-62/119

Normalised $X_1^3+X_2^3+X_3^3+\cdots X_{n-1}^3=X_n^3$	$57^3-181^3+176^3-300^3+295^3-419^3+414^3-538^3+533^3-657^3+652^3-776^3+771^3-895^3+890^3-1014^3+1009^3-1133^3+1128^3-1252^3+1247^3-1371^3+1366^3-1490^3+1485^3-1609^3+1604^3-1728^3+1723^3-1847^3+1842^3-1966^3+1961^3-2085^3+2080^3-2204^3+2199^3-2323^3+2318^3-2442^3+2437^3-2561^3+2556^3-2680^3+2675^3-2799^3+2794^3-2918^3+2913^3-3037^3+3032^3-3156^3+3151^3-3275^3+3270^3-3394^3+3389^3-3513^3+3508^3-3632^3+3627^3-3751^3+3746^3-3870^3+3865^3-3989^3+3984^3-4108^3+4103^3-4227^3+4222^3-4346^3+4341^3-4465^3+4460^3-4584^3+4579^3-4703^3+4698^3-4822^3-62^3+5888^3+4936^3-5060^3+5055^3-5179^3+5174^3-5298^3+5293^3-5417^3+5412^3-5536^3+5531^3-5655^3+5650^3-5774^3+5769^3-5893^3-186^3=-72^3$

— Parametrisation for determining cubes that sums up to another cube

Parameterisation for determining cubes that sums up to another cube has already been given by equations (4), (5), (6), (9), (10), (11), (17), (18) and (19). However these are again taken up for convenience. When number of terms n are four, equation of parametrisation is

$$\left\{\frac{a_1(A_3^3+A_2^3-A_4^3)}{3.(a_1.A_4^2+a_2.A_3^2-a_2.A_2^2).}\right\}^3+\left\{\frac{a_2.(A_3^3+A_2^3-A_4^3)}{3.(a_1.A_4^2+a_2.A_3^2-a_2.A_2^2).}+A_2\right\}^3$$
$$+\left\{A_3-\frac{a_2(A_3^3+A_2^3-A_4^3)}{3.(a_1.A_4^2+a_2.A_3^2-a_2.A_2^2).}\right\}^3=\left\{\frac{a_1(A_3^3+A_2^3-A_4^3)}{3.(a_1.A_4^2+a_2.A_3^2-a_2.A_2^2).}+A_4\right\}^3$$

where $A_4=\frac{a_2^2}{a_1^2}.(A_2+A_3)$ and $x=\frac{A_3^3+A_2^3-A_4^3}{3.\{a_1.A_4^2+a_2.(A_3^2-A_2^2)\}.}$

— On putting $A_3=0$ and $a_2=a_1.\beta$ then $A_4=A_2.\beta^2$ and $x=\frac{1-\beta^6}{3a_1.(\beta^4-\beta)}=-\frac{(\beta^3+1)}{3.a_1.\beta}$. On simplification of equation (6),

$$[-(1+\beta^3)]^3+[\beta.(2-\beta^3)]^3+[\beta.(1+\beta^3]^3=[(2.\beta^3-1)]^3 \qquad (20)$$

At $\beta=2$, $-9^3-12^3+18^3=15^3$. On dividing by common factor 3^3, $-3^3-4^3+6^3=5^3$.
At $\beta=-2$, $7^3-20^3+14^3=-17^3$
At $\beta=\frac{1}{3}$, $-84^3+53^3+28^3=-75^3$

— On putting $A_2+A_3=2.y$, $A_2-A_3=2.z$, $a_1=a$, $a_2=a.\beta$. That makes $A_4=2.y.\beta^2$ On simplifying, we obtain parametrisation with three variables y, z and β as given below.

$$[\{y^2(1-4.\beta^6)+3.z^2\}]^3+[\beta.\{y^2(1-4.\beta^6+6.\beta^3)+6.y.z.(\beta^3-1)-3.z^2\}]^3$$
$$+[\beta.\{y^2.(6.\beta^3-1+4.\beta^6)-6.y.z(\beta^3+1)+3.z^2\}]^3$$
$$=[\{y^2.(1+8.\beta^6)-12.\beta^3.y.z+3.z^2\}]^3 \qquad (21)$$

At y = 1, z = 2, $\beta=2$, $-243^3-270^3+414^3=333^3$
At y = -1, z = 2, $\beta=-2$, $-243^3-606^3+846^3=717^3$
At y = 3, z = 2, $\beta=2$, $-2283^3-3246^3+4830^3=4053^3$

— On putting z = 0 in equation (21), we obtain parametrisation with one variable β,

$$[(1-4.\beta^6)]^3+\left[-\beta.\left\{\left(2.\beta^3-\frac{3}{2}\right)^2-\frac{13}{4}\right\}\right]^3+\left[\beta.\left\{\left(2.\beta^3+\frac{3}{2}\right)^2-\frac{13}{4}\right\}\right]^3=[(1+8.\beta^6)]^3 \qquad (22)$$

At y $\beta=2$, $-255^3-414^3+606^3=513^3$
At y $\beta=3$, $-2915^3-8259^3+9231^3=5833^3$
At y $\beta=3/2$, $-1426^3-1167^3+3111^3=2948^3$

— On putting $\frac{y}{z}=s$ in equation (21), we obtain parametrisation with two variables s and β,

$$\left[\left\{s.(1-4.\beta^6)+\frac{3}{s}\right\}\right]^3+\left[\beta.\left\{s.(1-4.\beta^6+6.\beta^3)+6.(\beta^3-1)-\frac{3}{s}\right\}\right]^3$$
$$+\left[\beta.\left\{s.(6.\beta^3-1+4.\beta^6)-6(\beta^3+1)+\frac{3}{s}\right\}\right]^3=\left[\left\{s.(1+8.\beta^6)-12.\beta^3+\frac{3}{s}\right\}\right]^3 \qquad (23)$$

At s = 3, $\beta=2$, $-764^3-1160^3+1712^3=1444^3$
At s = 3, $\beta=-2$, $-764^3+1928^3-1328^3=1636^3$
At s = 4, $\beta=-2$, $-\left(\frac{4077}{4}\right)^3+\left(\frac{10134}{4}\right)^3-\left(\frac{6966}{4}\right)^3=\left(\frac{8595}{4}\right)^3$
or $-4077^3+10134^3-6966^3=8595^3$

— On putting $s=3$, we obtain parametrisation with one variable β,

$$\left[(-3).\left\{(\beta^3-1)(\beta^3+1)+\frac{2}{3}\right\}\right]^3+\left[(-3.\beta).\left\{(\beta^3-1)^2-\frac{2}{3}\right\}\right]^3+\left[3.\beta\left\{\left(\beta^3+\frac{1}{2}\right)^2-\frac{11}{12}\right\}\right]^3$$
$$=\left[6.\left\{\left(\beta^3-\frac{1}{4}\right)^2+\frac{5}{48}\right\}\right]^3 \qquad (24)$$

At $\beta = 2$, $-191^3 - 290^3 + 428^3 = 361^3$
At $\beta = -2$, $-191^3 + 482^3 + 332^3 = 409^3$
At $\beta = -3$, $-2186^3 + 7050^3 - 6312^3 = 4456^3$

— On putting $s = \beta$, we obtain parametrisation again with one variable β,

$$[\{\beta^2.(1 - 4.\beta^6) + 3\}]^3 + [\beta.\{\beta^2.(1 - 4.\beta^6 + 6.\beta^3) + 6.\beta.(\beta^3 - 1) - 3\}]^3 + [\beta.\{\beta^2.(6.\beta^3 - 1 + 4.\beta^6) - 6.\beta.(\beta^3 + 1) + 3\}]^3 = \left[\left\{\beta^2.(1 + 8.\beta^6) - \frac{3}{2}.(1 + 8.\beta^4) + \frac{9}{2}\right\}\right]^3 \quad (25)$$

At $\beta = 2$, $-1017^3 - 1494^3 + 2214^3 = 1863^3$
At $\beta = 3$, $-26232^3 - 72936^3 + 81576^3 = 51528^3$
At $\beta = 4$, $-262125^3 - 1017900^3 + 1066860^3 = 521235^3$

» Parametrisation Using Seed Equations

Using seed equation given in Table 4 at at n = 4 that is $3^3 + 4^3 + 5^3 = 6^3$ which can be written

$$(3.x + a_{31})^3 + (4.x + a_{41})^3 + (5.x + a_{51})^3 = (6.x + a)^3$$

where $a = \frac{3^2.a_{31} + 4^2.a_{41} + 5^2.a_{51}}{6^2}$ and $x = \frac{a^3 - a_{31}^3 - a_{41}^3 - a_{51}^3}{3.(3.a_{31}^2 + 4.a_{41}^2 + 5.a_{51}^2 - 6.a^2)}$.

Assuming $a_{31} = 0$, $a_{41} = A$, $a_{51} = \beta.A$ then $a = \frac{A.4^2 + A.\beta.5^2}{6^2}$. On putting these values in above equation and after simplification, we get parametrisation,

$$[-93093.\beta^3 + 90000.\beta^2 + 57600.\beta - 127680]^3 + [-124124.\beta^3 + 414840.\beta^2 - 441600.\beta + 223744]^3 + [139685.\beta^3 - 368400.\beta^2 + 489984.\beta - 212800]^3 = [18564.\beta^3 - 48960.\beta^2 + 158400.\beta - 80256]^3 \quad (26)$$

At $\beta = 0$, $-127680^3 + 223744^3 - 212800^3 = -80256^3$
and dividing by common factor 1216^3, $-105^3 + 184^3 - 175^3 = 66^3$
At $\beta = 1$, $-73173^3 + 72860^3 + 48469^3 = 47748^3$
At $\beta = -1$, $-2187^3 + 1204308^3 - 1210869^3 = -306180^3$
and dividing by common factor 729^3, $-3^3 + 1652^3 - 1661^3 = -420^3$
At $\beta = 1/2$, $-704133^3 + 729108^3 - 339579^3 = -87804^3$
and dividing by common factor 27^3, $-26079^3 + 27004^3 - 12577^3 = -3252^3$
At $\beta = -1/2$, $-978747^3 + 4510156^3 - 4538821^3 = -1392132^3$
and dividing by common factor 7^3, $-139821^3 + 644308^3 - 648403^3 = -198876^3$

In this way, there can be infinite parametrisation depending upon assignment of values by us.

» Parametrisation when number of terms are five

For obtaining parametrisation when number of terms are five, that is $n = 5$, we will have to use corresponding seed equation $1^3 + 1^3 + 5^3 + 6^3 = 7^3$ from Table 4. This seed equation will populate

$$(1.x + a_{11})^3 + (1.x + a_{12}.x)^3 + (5.x + a_{51})^3 + (6.x + a_{61}) = (7.x + a)^3$$

where $a = \frac{1^2.a_{11} + 1^2.a_{12} + 5^2.a_{51} + 6^2.A.\beta}{7^2}$ and $x = \frac{a^3 - (a_{11}^3 + a_{12}^3 + a_{51}^3 + a_{61}^3)}{3.(1.a_{11}^2 + 1a_{12}^2 + 5.a_{51}^2 + 6.a_{61}^2 - 7.a^2)}$

Let $a_{11} = 0$, $a_{12} = A$, $a_{51} = 0$, $a_{61} = A.\beta$, then

$a = \frac{1^2.a_{11}. + 1^2.a_{12} + 5^2.a_{51} + 6^2.A.\beta}{7^2} = A.\left(\frac{1}{7^2} + \beta.\frac{6^2}{7^2}\right)$ and $x = A.\frac{\left[\left(\frac{1}{7^2} + \beta.\frac{6^2}{7^2}\right)^3 - (1 + \beta^3)\right]}{3\left[1 + 6.\beta^2 - 7.\left(\frac{1}{7^2} + \beta.\frac{6^2}{7^2}\right)^2\right]}$. Putting values of 'a', x, $a_{11} = 0$, $a_{51} = 0$, $a_{61} = a_{12}.\beta$ in above populated equation, we get

$$\left[\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right)^3 - (1 + \beta^3)\right]^3 + \left[\left(\frac{1}{7^2} + \frac{6^2}{7^2}\beta\right)^3 - 21.\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right)^2 - \beta^3 + 18.\beta^2 + 2.\right]^3 + \left[5.\left\{\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right)^3 - \beta^3 - 1\right\}\right]^3 + \left[\left\{6.\left(\frac{1}{7^2} + \frac{6^2}{7^2}\beta\right)^3 - 21.\beta.\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right)^2 + 12.\beta^3 + 3.\beta - 6\right\}\right]^3 = \left[\left\{-14.\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right)^3 + 3.\left(\frac{1}{7^2} + \frac{6^2}{7^2}.\beta\right).(1 + 6.\beta^2) - 7.\beta^3 - 7\right\}\right]^3 \quad (27)$$

At x =2, $-669824^3 + 2670310^3 - 3349120^3 + 2661324^3 = 287350^3$
dividing by common factor, $-334912^3 + 1335155^3 - 1674560^3 + 1330662^3 = 143675^3$.
At x =-2, $-465632^3 + 4102118^3 - 2328160^3 + 4479180^3 = -2009770^3$
dividing by common factor, $-232816^3 + 2051059^3 - 1164080^3 + 2239590^3 = 1004885^3$.
At x =0, $-117648^3 + 234270^3 - 588240^3 - 70588^3 = -816354^3$
dividing by common factor, $-3096^3 + 6165^3 - 15480^3 - 18576^3 = -21483^3$

» Generalised Form when n is even integer

$$(a_1.x+A_1)^3+(a_2.x+A_2)^3+(A_3-a_2.x)^3+(a_4.x+A_4)^3+(A_5-a_4.x)^3+\cdots+(A_{n-1}-a_{n-2}.x)^3=(a_1.x+A_n)^3 \quad (28)$$

where $A_n=A_1+\frac{\{a_2^2.(A_2+A_3)+a_4^2,(A_4+A_5)\}+a_6^2(A_6+A_7)+\cdots a_{n-2}^2.(A_{n-2}+A_{n-1})}{a_1^2}$

and $x=\frac{A_n^3-(A_1^3+A_2^3+A_3^3+\cdots+A_{n-1}^3)}{3.\{a_1.(A_1^2-A_n^2)+a_2(A_2^2-A_3^2)+\cdots+a_{n-2}.(A_{n-2}^2-A_{n-1}^2)\}}$

» Generalised Form when n is odd integer

As explained in paragraphs 2.2 and 2.2a,

$$\{(x+a_{11})^3+(x+a_{12})^3+(x+a_{13})+\ldots\text{upto}\,(x+a_{1x_1})\}+$$
$$\left\{(2.x+a_{21})^3+(2.x+a_{22})^3+(2.x+a_{23})+\cdots\text{upto}\,(2.x+a_{2x_2})^3\right\}+$$
$$\left\{(3.x+a_{31})^3+(3.x+a_{32})^3+(3.x+a_{33})^3+\ldots\text{upto}\,(3.x+a_{3x_3})^3\right\}+$$
$$\left\{(k.x+a_{k1})^3+(k.x+a_{k2})^3+(k.x+a_{k3})^3+\ldots\text{upto}\,(k.x+a_{kx_k})^3\right\}=\{(m+p).x+a\}^3 \quad (29)$$

where 'a' has value equal to

$$\frac{\{(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1})+2^2.(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2})+3^2.(a_{31}+a_{32}+a_{33}+\cdots+a_{3x_3})+\cdots+k^2.(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k})\}}{(m+p)^2}$$

and x has value equal to

$$\frac{\left[a^3-\{(a_{11}^3+a_{12}^3+a_{13}^3+\cdots+a_{1x_1}^3)+(a_{21}^3+a_{22}^3+a_{23}^3+\cdots+a_{2x_2}^3)+(a_{31}^3+a_{32}^3+a_{33}^3+\cdots+a_{3x_3}^3)+\cdots+(a_{k1}^3+a_{k2}^3+a_{k3}^3+\cdots a_{kx_k}^3)\}\right]}{3.\{(a_{11}^2+a_{12}^2+a_{13}^2+\cdots a_{1x_1}^2)+2.(a_{21}^2+a_{22}^2+a_{23}^2+\cdots a_{2x_2}^2)+3.(a_{31}^2+a_{32}^2+a_{33}^2+\cdots a_{3x_3}^2)+\cdots+k.(a_{k1}^2+a_{k2}^2+a_{k3}^2+\cdots a_{kx_k}^2)-(m+p).a^2\}}$$

and $\{(a_{11},a_{12},a_{13},\ldots a_{1x_1}),(a_{21},a_{22},a_{23},\ldots a_{2x_2}),(a_{31},a_{32},a_{33},\ldots a_{3x_3})\ldots(a_{k1},a_{k2},a_{k3},\ldots a_{kx_k})\}$, are real rational quantities assigned by us.

3. RESULTS AND CONCLUSIONS

Sum of $(n-1)$ cubes (each of different integer values) that equals another cube i.e. $X_1^3+X_2^3+X_3^3+\cdots X_{n-1}^3=X_n^3$ can be expressed as

$$(a_1.x+A_1)^3+(a_2.x+A_2)^3+(A_3-a_2.x)^3+(a_4.x+A_4)^3+(A_5-a_4.x)^3+\cdots+(A_{n-1}-a_{n-2}.x)^3=(a_1.x+A_n)^3.$$

When n is an even integer the terms containing x^3 cancel each other and we are left with a quadratic equation. Coefficient of x^2 is then equated to zero and it reduces the quadratic into a linear equation. In this way, a cubic reduced to a linear equation gives straight way the value of x as mentioned below

$$x=\frac{A_n^3-(A_1^3+A_2^3+A_3^3+\cdots+A_{n-1}^3)}{3.\{a_1.(A_1^2-A_n^2)+a_2(A_2^2-A_3^2)+\cdots+a_{n-2}.(A_{n-2}^2-A_{n-1}^2)\}}$$

$$\text{where } A_n=A_1+\frac{\{a_2^2.(A_2+A_3)+a_4^2,(A_4+A_5)\}+a_6^2(A_6+A_7)+\cdots a_{n-2}^2.(A_{n-2}+A_{n-1})}{a_1^2}$$

and once x and A_n are known, $a_2,a_4,a_6,..a_{n-2}$, $a_1,A_1,A_2,A_3,\ldots A_{n-1}$ have real and rational values assigned by us, therefore, values of $(a_1.x+A_1),(a_2.x+A_2),(A_3-a_2.x)\ldots(A_{n-1}-a_{n-2}.x),(a_1.x+A_n)$ can be computed and after multiplication with LCM, sum of cubes that equals another cube in integer form can be found. In nutshell, it can be stated that sum of cubes equalling another cube when its terms n are even and $n\geq 4$, can be written as a cubic equation which can be reduced to a linear equation by self cancellation of cubic terms and after equating coefficient of x^2 term to zero, gives value of x and hence values of cubes.

Above method is applicable when number of terms are even and since number of terms can not always be even, therefore, this method is not applicable to odd 'n' and another method is devised whereby a seed or empirical equation of sum of $(n-1)$ cubes that equals another cube is formulated

$$(m+p)^3=x_1.(1^3)+x_2.(2^3)+x_3.(3^3)+\cdots x_k.(k^3)$$

where $x_1=\{(n-1)-(x_2+x_3+x_4+\cdots+x_k)\}$ and can have any integer value including 0. And n is number of terms, m is an integer such that $n-1$ lies between $(m-1)^3$ and m^3, $(x_2+x_3+x_4+\cdots+x_k)\leq(n-1)$, p is any positive integer including zero and $x_2, x_3, \ldots x_k$ are positive integers including zero. Above said equation can be written as

$$(m+p)^3=[1^3+1^3+1^3+\cdots\text{repeated } x_1 \text{ times}]+(2^3+2^3+2^3+\cdots\text{repeated } x_2 \text{ times})+$$
$$(3^3+3^3+3^3+\cdots\text{repeated } x_3 \text{ times})+\cdots+(k^3+k^3+k^3+\cdots\text{repeated } x_k \text{ times})$$

Then

$$\{(x+a_{11})^3+(x+a_{12})^3+(x+a_{13})+\ldots\text{upto}\,(x+a_{1x_1})\}+$$
$$\left\{(2.x+a_{21})^3+(2.x+a_{22})^3+(2.x+a_{23})+\cdots\text{upto}\,(2.x+a_{2x_2})^3\right\}+$$
$$\left\{(3.x+a_{31})^3+(3.x+a_{32})^3+(3.x+a_{33})^3+\ldots\text{upto}\,(3.x+a_{3x_3})^3\right\}+$$
$$\left\{(k.x+a_{k1})^3+(k.x+a_{k2})^3+(k.x+a_{k3})^3+\ldots\text{upto}\,(k.x+a_{kx_k})^3\right\}=\{(m+p).x+a\}^3$$

and after normalisation gives integer value of sum of cubes equal to another cube provided 'a' has value equal to

$$\frac{\left\{(a_{11}+a_{12}+a_{13}+\cdots+a_{1x_1})+2^2.(a_{21}+a_{22}+a_{23}+\cdots+a_{2x_2})+3^2.(a_{31}+a_{32}+a_{33}+\cdots+a_{3x_3})+\cdots+k^2.(a_{k1}+a_{k2}+a_{k3}+\cdots+a_{kx_k})\right\}}{(m+p)^2}$$

and x has value equal to

$$\frac{\left[a^3-\left\{\begin{matrix}(a_{11}^3+a_{12}^3+a_{13}^3+\cdots+a_{1x_1}^3)+(a_{21}^3+a_{22}^3+a_{23}^3+\cdots+a_{2x_2}^3)\\+(a_{31}^3+a_{32}^3+a_{33}^3+\cdots+a_{3x_3}^3)+\cdots+(a_{k1}^3+a_{k2}^3+a_{k3}^3+\cdots a_{kx_k}^3)\end{matrix}\right\}\right]}{3.\left\{\begin{matrix}(a_{11}^2+a_{12}^2+a_{13}^2+\cdots a_{1x_1}^2)+2.(a_{21}^2+a_{22}^2+a_{23}^2+\cdots a_{2x_2}^2)+3.(a_{31}^2+a_{32}^2+a_{33}^2+\cdots a_{3x_3}^2)\\+\cdots+k.(a_{k1}^2+a_{k2}^2+a_{k3}^2+\cdots a_{kx_k}^2)-(m+p).a^2\end{matrix}\right\}}$$

where $\{(a_{11},a_{12},a_{13},\dots a_{1x_1}),(a_{21},a_{22},a_{23},\dots a_{2x_2}),(a_{31},\ a_{32},\ a_{33},\dots a_{3x_3})\dots(a_{k1},a_{k2},\ a_{k3},\dots a_{kx_k})\}$, are real rational quantities assigned by us, values of m and p are known from seed equation and value of ‘a’ is found by using equation (18). When x is determined according to equation (19), values of cubes are known after normalisation of equation (17) by multiplying with LCM. In nutshell, a seed equation is first determined and then that seed equation inherently develops a quadratic equation that is reduced to a linear equation by equating coefficient of x^2 to zero. Once x and ‘a’are determined, sum of cubes that equals another cube can be found after multiplying the terms with LCM.

Acknowledgements

I acknowledge the help provided by the wonderful website https://www.desmos.com in calculating the values of tedious exponential terms and successive multiplications of terms.

References

[1] Berndt, B. C. and Bhargava, S. "Ramanujan--For Lowbrows." Amer. Math. Monthly 100, 1993.

[2] Gardiner, V. L.; Lazarus, R. B.; and Stein, P. R. "Solutions of the Diophantine Equation $x^3+y^3=z^3-d$." Math. Comput. 18, 1964.

[3] Koyama, K.; Tsuruoka, Y.; and Sekigawa, H. "On Searching for Solutions of the Diophantine Equation $x^3+y^3+z^3=n$" Math. Comput. 66, 1997.

[4] Miller, J. C. P. and Woollett, M. F. C. "Solutions of the Diophantine Equation $x^3+y^3+z^3=k$" J. London Math. Soc. 30, 1955.

[5] Piezas, T. "Ramanujan and the Cubic Equation $3^3+4^3+5^3=6^3$." 2005, http://www.geocities.com/titus_piezas/RamCube.pdf

[6] Piezas, Tito III and Weisstein, Eric W. "Diophantine Equation--3rd Powers." From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/DiophantineEquation3rdPowers.html

ISSN 1584 - 2665 (printed version); ISSN 2601 - 2332 (online); ISSN-L 1584 - 2665