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LYAPUNOV STABILITY CONCEPTS IN MOVEMENT PHENOMENON STUDY

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ABSTRACT:
This study presents few theoretical considerations upon Lyapunov stability and presents a criterion of asymptotic stability for a linear oscillator with variable parameters. It is shown that this criterion is close to a necessary and sufficient conditions of asymptotic stability and the instability of the small oscillations is proved.

KEYWORDS:
stability, asymptotical stability, instability, equilibrium, Lyapunov

1. INTRODUCTION

Most of the theories examining a stability problem of the zero solution of a differential equation are based on the Lyapunov stability and instability theorems. They use a corresponding Lyapunov function [1] which is assumed as an energy type function like:

$$V = \frac{1}{2}c_1(t)\left(x'\right)^2 + \frac{1}{2}c_2(t)x^2 \tag{1}$$

where $c_1(t), c_2(t)$ are time variable functions.

For a better understanding of this study let's show a few considerations about what means a Lyapunov stability. Let us consider the equation

$$x' = v(x), \qquad x \in R^n \tag{2}$$

such that the zero solution is obtained for $v(0) = 0$. It has the solution $\varphi(t) \equiv 0$ for initial condition $\varphi(t_0) = 0$ and we will study the solutions with almost alike initial conditions.

Definition 1. The stability solution of equation (2) is called Lyapunov stable position [2] if for all $\varepsilon > 0$ exists $\delta > 0$ (which depends on ε only) such that for all x_0 with $\|x_0\| < \delta$, the solution φ has the initial condition $\varphi(0) = x_0$ and $\|\varphi(t)\| < \varepsilon$ for all $t > 0$.

Definition 2. The equilibrium position $x = 0$ of equation (2) is called asymptotical stable [2] if it is Lyapunov stable and for all solutions φ with initial condition $\varphi(0) = x_0$, $\lim\limits_{t\to\infty}\varphi(t) = 0$ is fulfilled (Figure 1).

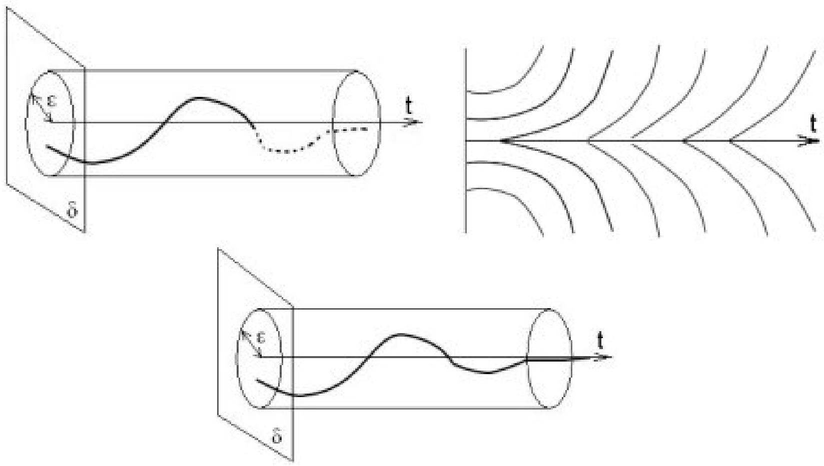

FIGURE 1. The stable, unstable and asymptotical stable equilibrium position

Remark 1.

If $x \in R^n, x = (x_1, \dots, x_n)$ then $\|x\|^2 = x_1^{\ 2} + \dots + x_n^{\ 2}$.

Let us consider an oscillator described by the following differential equation:

$$x'' + f(t)x' + g(t)x = 0 \qquad (3)$$

where the dumping and rigidity coefficients f(t) and g(t) are continuous and bounded functions of the time t.

2. ASYMPTOTICAL STABILITY OF ZERO SOLUTION

Assuming that g(t) is continuously, differentiable and that the inequalities:

$$|f(t)| < M_1 \quad , \quad |g(t)| < M_2 \quad , \quad |g'(t)| < M_3 \qquad (4)$$

hold for $t \in R_+ = [0, \infty)$ we have:

Theorem 1.

If the conditions

$$g(t) > \alpha_1 > 0\,, \quad p(t) = \frac{1}{2}\frac{g'(t)}{g(t)} + f(t) > \alpha_2 > 0 \qquad (5)$$

are fulfilled, then the solution

$$x = 0 \quad , \quad x' = 0 \qquad (6)$$

of differential equation (3) is asymptotically stable.

Proof:

If we consider the function

$$V_1 = \frac{1}{2}\left(x^2 + 2\beta\frac{xx'}{\sqrt{g(t)}} + \frac{(x')^2}{g(t)}\right) \qquad (7)$$

with $\beta = \text{constant}$, its time derivative has the form

$$V_1' = \frac{1}{\sqrt{g(t)}}\left[\left(-\frac{p(t)}{\sqrt{g(t)}} + \beta\right)(x')^2 - \beta p(t)xx' - \beta g(t)x^2\right] \qquad (8)$$

If we take $\beta > 0$ sufficiently small, then $V_1 > 0, V_1' < 0$. We can take $0 < \beta < \min\left\{1, \frac{\alpha_2}{2\sqrt{M_2}}, \frac{8\alpha_1^{\ 3}\alpha_2}{(2\alpha_1 M_1 + M_3)^2} \cdot \frac{1}{\sqrt{M_2}}\right\}$. Thus all conditions of Lyapunov theorem are fulfilled and the zero solution of equation (3) is asymptotically stable.

3. INSTABILITY OF THE ZERO SOLUTION

In order to obtain instability conditions of the zero solution, we note $x' = y$, then we get the system:

$$\begin{cases} x' = y \\ y' = -g(t)x - f(t)y \end{cases} \tag{9}$$

which is equivalent to equation (3). It has the trivial solution:

$$x = 0 \quad , \quad y = 0 \tag{10}$$

In [3], A.O. Ignatyev obtained some criterion of asymptotic stability of solution (6) of the equation (3).

Theorem 2.
The solution (10) of the system (9) is unstable if exists t_0 such that, for each $t > t_0$ is fulfilled one of the following conditions:

$$D(t) = \frac{1}{4}f^2(t) + g(t) \le 0 \tag{11}$$

$$D(t) > 0\,, 4f(t)D(t) + \frac{1}{2}f'(t)f(t) + g'(t) - \left[f'(t) + f^2(t) + 4D(t)\right]\sqrt{D(t)} < 0 \tag{12}$$

$$D(t) > 0\,, 4f(t)D(t) + \frac{1}{2}f'(t)f(t) + g'(t) + \left[f'(t) + f^2(t) + 4D(t)\right]\sqrt{D(t)} < 0 \tag{13}$$

Proof:
If $\varepsilon > 0$ is an arbitrary positive number our goal is to show that for any sufficiently small $\delta > 0$ exist x_0, y_0 with

$$|x_0| < \delta, \quad |y_0| < \delta \tag{14}$$

and exists T>0 such that, for $t = t_0 + T$ the trajectory $x(t), y(t)$ $(x(t_0) = x_0\,, y(t_0) = y_0)$ reaches the boundary of the domain

$$|x| < \varepsilon \quad , \quad |y| < \varepsilon \tag{15}$$

If consider the function $V = xy$, its time derivative has the form $V' = y^2 - f(t)xy - g(t)x^2$. Now we take $x_0 > 0, y_0 > 0$ satisfying (14) such that $V'(t_0, x_0, y_0) = y_0{}^2 - f(t_0)x_0y_0 - g(t_0)x_0{}^2 > 0$. Considering the trajectory $x(t), y(t)$ of the system (9) with initial data $x(t_0) = x_0\,, y(t_0) = y_0$ we can assume $D(t_0) < 0$. Let $[t_0, t_1]; [t_2, t_3]; \ldots; [t_{2n}, t_{2n+1}]; \ldots$ be segments on which condition (11) holds and $(t_1, t_2); (t_3, t_4); \ldots; (t_{2n-1}, t_{2n}); \ldots$ be intervals on which conditions (12) or (13) are valid. As $V' \ge 0$ on $[t_0, t_1]$, the trajectory is staying in the domain $xy \ge x_0y_0$ on this segment. If $t \in (t_1, t_2)$ on this interval V' changes its sign. But $V' = 0$ only if one of the following are valid:

$$y = x\left(\frac{1}{2}f(t) + \sqrt{D(t)}\right) \tag{16}$$

or

$$y = x\left(\frac{1}{2}f(t) - \sqrt{D(t)}\right) \tag{17}$$

And $V' > 0$ if

$$y \rangle x\left(\frac{1}{2}f(t) + \sqrt{D(t)}\right) \tag{18}$$

or

$$y < x\left(\frac{1}{2}f(t) - \sqrt{D(t)}\right) \tag{19}$$

Let $t^* \in (t_1, t_2)$ be such moment of time, that $y(t^*) = x(t^*)\left[\frac{1}{2}f(t^*) + \sqrt{D(t^*)}\right]$ and that means that the point of the trajectory belongs to the straight line (13) for $t = t^*$. Under the condition $V' = 0$, V'' has the expression:

$$V'' = -x^2\left[4f(t)D(t) + \frac{1}{2}f'(t)f(t) + g'(t) + \left(f'(t) + f^2(t) + 4D(t)\right)\sqrt{D(t)}\right] \tag{20}$$

Because of (13) we observe that $V'' > 0$ for $V' = 0$, that means that the trajectory belongs to the domain $V' > 0$ when $t \in (t^*, t^* + \Delta t)$.

Let $t^{**} \in (t_1, t_2)$ be such moment of time, that $y(t^{**}) = x(t^{**})\left[\frac{1}{2}f(t^{**}) + \sqrt{D(t^{**})}\right]$ and that means that the point of the trajectory belongs to the straight line (17) for $t = t^{**}$. Because of (12) we can observe that $x(t), y(t)$ are satisfying relation (19), for $t \in (t^{**}, t^{**} + \Delta t)$. This means that the trajectory belongs to the domain $V' \geq 0$ when $t \in [t_1, t_2]$.

We can show analogously that the point $x(t), y(t)$ belongs to the set $V' \geq 0$ when $t \in [t_n, t_{n+1}]$ $(n = 3,4,\ldots)$ It means that the inequality $V'(x(t), y(t)) \geq 0$ holds for every $t \geq t_0$ [4]. And further means that $x(t)y(t) \geq x_0 y_0$ for every $t \geq t_0$.

In [5], N.N. Moiseyev wrote a differential equation of small plane oscillations of a rocket, whose centre of mass moves rectilinearly with constant velocity. The author obtained sufficient conditions for stability of zero solution and showed that in case of plane small oscillations of a rocket these conditions are not fulfilled.

In conclusion, the boundary of the domain (15) is reached for the finite interval of time. Applying Theorem 2 we can conclude that exists $t_0 > 0$ such that the inequalities (13) holds for any $t \geq t_0$. With this considerations we prove the instability of the small oscillations.

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