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TWO COMPUTATIONAL METHODS USING THE CHEBYSHEV APPROXIMATION

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ABSTRACT:
This paper presents a technique that is frequently used in evaluating functions, the Chebyshev approximation of a real, one variable function. The theoretical considerations conclude that the polynomial of best approximation is difficult to find, but the Chebyshev approximation, or the minimax polynomial which approximates the function is almost identical and easy to compute. A numerical example computed in MathCAD is considered, showing two methods finding the minimax polynomial.
KEYWORDS:
evaluating functions, minimax polynomial, Chebyshev approximation

1. INTRODUCTION

The purpose of this study is to present a technique that is frequently used in evaluating functions. In the Section 2 we will discuss about the Chebyshev approximation of a real, one variable function and two methods to find the better Chebyshev approximating polynomial are depicted. Because the smooth spreading of the error is a very important property in evaluating functions, the Chebyshev approximation is very nearly the minimax polynomial, which among all polynomials of the same degree has the smallest maximum deviation from the true function. The minimax polynomial is difficult to find in practical cases, thus the Chebyshev approximating polynomial is almost identical and easy to compute. The Section 3 presents a numerical example computed in MathCAD and the Section 4 concludes the paper.

2. METHODOLOGY

Let $x \in [-1, 1]$ and the variable $\theta \in [0, \pi]$ denoted by the relations

$$x = \cos\theta \ , \ \theta = \arccos x \tag{1}$$

The Chebyshev polynomial of degree n is denoted $T_n(x)$ an is given by the explicit formula

$$T_n(x) = \cos n\theta \Leftrightarrow T_n(x) = \cos(n \arccos x) \tag{2}$$

In this interval (-1, 1) the polynomial (2) has n zeros, and they are located at the points

$$T_n(x) = \cos n\theta = 0 \Rightarrow \theta_k = (2k-1)\pi/2n, \quad k \in \overline{1,n} \ ,$$
$$x_k = \cos\theta_k = \cos[(2k-1)\pi/2n], \quad k \in \overline{1,n} \tag{3}$$

In the same interval there are n+1 extrema of the function $T_n(x)$, located at

$$x = \cos\left(\frac{\pi k}{n}\right) \quad k = 0, 1, \ldots, n \tag{4}$$

The Chebyshev polynomials are orthogonal in the considered interval (-1, 1) over a weight $\left(1 - x^2\right)^{-1/2}$. In particular we observe that

$$\int_{-1}^{1} \frac{T_i(x) T_j(x)}{\sqrt{1 - x^2}} dx = \begin{cases} 0 & i \neq j \\ \pi / 2 & i = j \neq 0 \\ \pi & i = j = 0 \end{cases} \tag{5}$$

The Chebyshev polynomials satisfy a discrete orthogonality relation as well the continuous one (5) and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k) T_j(x_k) = \begin{cases} 0 & i \neq j \\ m/2 & i = j \neq 0 \\ m & i = j = 0 \end{cases} \tag{6}$$

Relation (2) can be combined with the next trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos(n-1)\theta \cdot \cos\theta \tag{7}$$

to yield a recurrence for $T_n(x)$:

$$T_n(x) = 2x T_{n-1}(x) - T_{n-2}(x) \tag{8}$$

There is a close relation between the Chebyshev polynomials and the discrete Fourier transform, thus the Chebyshev approximating polynomial for a function defined on (-1, 1) is:

$$f(x) = \sum_{p=0}^{\infty} a_p T_p(x) \tag{9}$$

where

$$a_p = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_p(x)}{\sqrt{1 - x^2}} dx \quad , \qquad a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx \tag{10}$$

These relations can be obtained from the Fourier series expansion

$$f(\cos\theta) = \sum_{p=0}^{\infty} a_p \cos p\theta \tag{11}$$

having the coefficients

$$a_p = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \cos p\theta \cdot d\theta \,, \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(\cos\theta) d\theta \tag{12}$$

In some practical cases the Chebyshev polynomials can be generated by the function

$$\frac{1 - tx}{1 - 2tx + t^2} = 1 + T_1(x) + t^2 T_2(x) + \ldots \tag{13}$$

It is interesting to observe from (8) that, the coefficient of the leading term x^n of the $T_n(x)$ is 2^{n-1}, thus the polynomial $\overline{T}_n(x) \equiv 2^{1-n} T_n(x)$ has the coefficient of x^n equal to one. A polynomial whose term of highest degree has a coefficient of 1 is called "monic". The recurrence for the Chebyshev monic polynomials exists in consequence of (8):

$$\overline{T}_n(x) = x\overline{T}_{n-1}(x) + \frac{1}{4}\overline{T}_{n-2}(x) \tag{14}$$

The Chebyshev monic polynomials have the same zeros like the common Chebyshev polynomials, but the extrema are different:

$$\bar{T}_n(x_p) = \frac{(-1)^p}{2^{n-1}} \quad \text{for } x_p = \cos\frac{p\pi}{n} \quad , \quad p = \overline{0,n} \tag{15}$$

The property that makes the Chebyshev polynomials so useful in polynomial approximation of functions is that among all monic polynomials of degree n, defined on (-1, 1), the Chebyshev monic polynomial has the minimal ∞-norm. For every continuous function $f \in C([a,b])$, the ∞-norm is denoted by

$$\| f \| = \max_{x\in[a,b]} | f(x) | \tag{16}$$

We denote, as well, by $p_n^*(x) \in \Pi_n$, the minimax polynomial of a function $f \in C([a,b])$, otherwise speaking

$$\left\| f - p_n^* \right\| = \min_{p_n\in\Pi_n} \| f - p_n \| = \min_{p_n\in\Pi_n} \max_{x\in[a,b]} | f(x) - p_n(x) | \tag{17}$$

With relations (2), (3) and (6) the next result occurs:

If f(x) is an arbitrary function in the interval (-1, 1) and if n coefficients c_j , j = 1, 2, ..., n are defined by

$$c_j = \frac{2}{n}\sum_{k=1}^{n} f(x_k)T_{j-1}(x_k) = \frac{2}{n}\sum_{k=1}^{n} f\left[\cos\left(\frac{\pi(2k-1)}{2n}\right)\right]\cos\left(\frac{\pi(j-1)(2k-1)}{2n}\right) \tag{18}$$

then the approximation formula

$$f(x) \approx \left[\sum_{k=1}^{n} c_k T_{k-1}(x)\right] - \frac{1}{2}c_1 \tag{19}$$

is exact for x equal to all of the n zeros of $T_n(x)$.

Another method is to expand the function in a Taylor series

$$P_n(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) \tag{20}$$

having the approximation rest

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1} \tag{21}$$

The Taylor polynomials are precise in close proximity of x_0, but they are worthless for high difference between x and x_0. Because the Chebyshev polynomials have a smooth spreading of the error, they can be useful to reduce the Taylor polynomials order. In order to attain the purpose, we shall replace x^n of polynomial P_n with a Chebyshev polynomials combination and we shall then exclude the term including $T_n(x)$.

3. EXAMPLE AND SOLVING

Let f(x) = cos(x)/(1+e^x) be the considered function. The next Mathcad application will find the Chebyshev approximating polynomial for f(x). First, the function will be expanded into a MacLaurin series of 5 grade, the term x^5 will be then replaced with a Chebyshev polynomials combination. The term containing Chebyshev $T_5(x)$ will be eluded.

$\mathrm{ORIGIN} \equiv 1$

$f(x) := \dfrac{\cos(x)}{1+e^{x}} \quad x0 := 0$

$i := 1 .. 5$

$d_i := \dfrac{d^{i}}{dx0^{i}} f(x0) \qquad d = \blacksquare$

$$\mathrm{MacLaurin}(x) := f(x0) + \frac{x-x0}{1!}\cdot d_1 + \frac{(x-x0)^2}{2!}\cdot d_2 + \frac{(x-x0)^3}{3!}\cdot d_3 + \frac{(x-x0)^4}{4!}\cdot d_4 + \frac{(x-x0)^5}{5!}\cdot d_5$$

$\mathrm{MacLaurin}(x) \rightarrow$

$\mathrm{T\_maxim}(x) := x^{5}$

$$\mathrm{ceb}(x,n) := \begin{array}{|l} 1 \quad \text{if } n = 0 \\ x \quad \text{if } n = 1 \\ (2\cdot x\cdot \mathrm{ceb}(x,n-1)) - \mathrm{ceb}(x,n-2) \quad \text{otherwise} \end{array}$$

$\mathrm{ceb}(x,0) \rightarrow$

$\mathrm{ceb}(x,1) \rightarrow$

$\mathrm{ceb}(x,2) \rightarrow$

$\mathrm{ceb}(x,3) \rightarrow$

$\mathrm{ceb}(x,4) \rightarrow$

$\mathrm{ceb}(x,5) \rightarrow$

$a0 := \dfrac{1}{\pi}\cdot \displaystyle\int_{-1}^{1} \frac{x^{5}}{\sqrt{1-x^{2}}}\,dx \qquad a0 = \blacksquare$

$i := 1 .. 5$

$a_i := \dfrac{2}{\pi}\cdot \displaystyle\int_{-1}^{1} \frac{x^{5}\cdot \mathrm{ceb}(x,i)}{\sqrt{1-x^{2}}}\,dx \qquad a = \blacksquare$

$$\mathrm{T\_maxim}(x) := a0\cdot \mathrm{ceb}(x,0) + a_1\cdot \mathrm{ceb}(x,1) + a_2\cdot \mathrm{ceb}(x,2) + a_3\cdot \mathrm{ceb}(x,3) + a_4\cdot \mathrm{ceb}(x,4) + a_5\cdot \mathrm{ceb}(x,5)$$

$$\mathrm{T\_maxim}(x) := a0\cdot \mathrm{ceb}(x,0) + a_1\cdot \mathrm{ceb}(x,1) + a_2\cdot \mathrm{ceb}(x,2) + a_3\cdot \mathrm{ceb}(x,3) + a_4\cdot \mathrm{ceb}(x,4)$$

$\mathrm{T\_maxim}(x) \rightarrow$

$$\mathrm{Cebasev}(x) := \frac{1}{2} - \frac{1}{4}\cdot x - \frac{1}{4}\cdot x^{2} + \frac{7}{48}\cdot x^{3} + \frac{1}{48}\cdot x^{4} - \frac{11}{480}\cdot \mathrm{T\_maxim}(x)$$

$\mathrm{Cebasev}(x) \rightarrow$

Having the expansion in MacLaurin series of 5 grade, a Chebyshev approximating polynomial of 4 grade was computed. In Figure 1 the graphs of function f(x), the MacLaurin approximating polynomial and the Chebyshev approximating polynomial are shown in comparison. Obviously, the approximation works in the close proximity of zero, with a high error outside the interval (-1, 1).

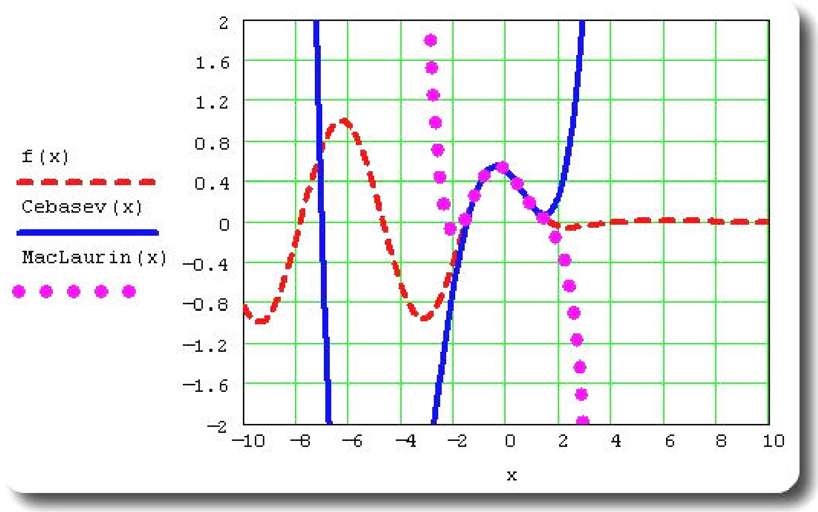

Fig. 1. The graphs of function f(x), the expansion in MacLaurin series and the Chebyshev approximating polynomial are shown in comparison

The next secuence is relevant to observe that the difference between f(x) and its Chebyshev approximating polynomial has a low acceptable value in the close proximity of zero.

$j := 1 .. 15 \qquad a_j := -4 + 0.57(j-1) \qquad error_j := \left|f(a_j) - Cebasev(a_j)\right|$

$error^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4.053	2.525	1.289	0.491	0.11	$401\cdot10^{-3}$	$033\cdot10^{-5}$	$159\cdot10^{-5}$	$218\cdot10^{-4}$	$251\cdot10^{-3}$	0.134	0.645	1.886	4.259	8.201

The second method to compute the minimax polynomial that approximates the considered function is using formulae (18) and (19). The next MathCad code was developed.

$n := 10$

$j := 1 .. n$

$$c_j := \frac{2}{n} \cdot \sum_{k=1}^{n} f\left[\cos\left[\frac{\pi\cdot(2\cdot k-1)}{2\cdot n}\right]\right]\cdot\cos\left[\frac{\pi\cdot(j-1)(2\cdot k-1)}{2\cdot n}\right]$$

$$minimax(x) := \sum_{k=1}^{n} c_k\cdot ceb(x, k-1) - \frac{1}{2}c_1$$

In Figure 2 the graphs of function f(x), the Chebyshev approximating polynomial computed with the first method and the minimax approximating polynomial as it was computed with the second method are depicted in comparison.

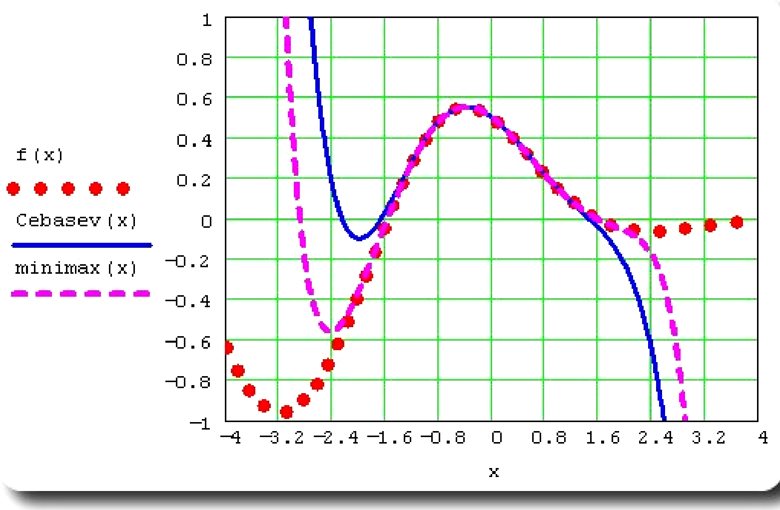

Fig. 2. The graphs of function f(x), the Chebyshev approximating polynomial computed with the first method and the minimax approximating polynomial computed with the second method

It is obviously that the new minimax approximating polynomial is better than the first Chebyshev approximating polynomial, having a larger interval where the difference between the function value and minimax polynomial value has an acceptable score.

4. CONCLUSIONS

The Chebyshev approximating polynomials are more efficient in real function approximations and they make a well description of function variations on interval (-1, 1). The advantage of using them in approximation theory is that we can have an acceptable error computing a less grade polynomial instead of other's approximation kinds. One particular application of Chebyshev methods is the economization of power series, which is also o useful technique in approximating functions.

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