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CONSIDERATIONS UPON THE VON MISES 2-DIMENSIONAL DISTRIBUTION

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ABSTRACT:
In this paper we generalyzed the one-dimensional von Mises distribution at 2-dimensional space, by giving its expresion. We showed that this 2D rule meets all conditions to be a continuous distribution density, and it can be used in future modeling of certain statistical problems.
KEYWORDS:
Continuous distribution density, von Mises distribution, Data modeling

1. INTRODUCTION

The aim of this paper is to generalyze the one-dimensional von Mises distribution at 2-dimensional space. We present in the Section 2 the expresion of 2-D Mises distribution that we propose and the condition that this function has to meet in order to be a continuous distribution density. In order to verify that the giving expresion is an efficient one, we considered two sets of input data, exposed in Section 3. Calculations have been done in MathCad. Section 4 includes the conclusions of the paper.

2. METHODOLOGY

In probability theory and statistics, the von Mises distribution (also known as the circular normal distribution) is a continuous probability distribution (2, 5). It may be thought of as the circular version of the normal distribution, since it describes the distribution of a random variate with period 2π. It is used in applications of directional statistics, where a distribution of angles are found which are the result of the addition of many small independent angular deviations, such as target sensing, or grain orientation in a granular material. If *x* is the angular random variable, it is often useful to think of the von Mises distribution as a distribution of complex numbers $z = e^{ix}$ rather than the real numbers *x* (3).

The von Mises probability density function (2, 4) for the angle *x* is given by:

$$f(x) = \frac{e^{k\cos(x-\mu)}}{2\pi I_0(k)} \tag{1}$$

where $I_0(x)$ is the modified Bessel function of order 0. The parameters μ and k>0 can be understood by considering the case where k is large. Our aim in this paper is to generalyze the one-dimensional von Mises distribution at 2-dimensional space.

Let $f:[\mu_1-\pi,\mu_1+\pi]\times[\mu_2-\pi,\mu_2+\pi]\rightarrow R$ be the 2-dimensional von Mises distribution rule, expressed by

$$f(x,y)=\frac{e^{k_1\cos(x-\mu_1)}}{2\pi I_0(k_1)}\cdot\frac{e^{k_2\cos(x-\mu_2)}}{2\pi I_0(k_2)} \tag{2}$$

that is

$$f(x,y)=\frac{1}{4\pi^2 I_0(k_1)I_0(k_2)}e^{k_1\cos(x-\mu_1)+k_2\cos(x-\mu_2)} \tag{3}$$

x and y are the independent variables, parameters $k_1>0$, $k_2>0$, μ_1, μ_2 are real constants and $I_0(t)$ is the modified Bessel function of first kind, having the form

$$I_0(t)=\sum_{p=0}^{10}\frac{\left(\frac{t}{2}\right)^{0+2p}}{p!\cdot\Gamma(0+p+1)} \tag{4}$$

In order to meet our purpose, function $f(x,y)$ has to be a probability density (1, 6), and therefore has to meet the conditions

$$f(x,y)\geq 0 \tag{5}$$

and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x,y)dxdy=1 \tag{6}$$

Now we are going to define the 2-dimensional von Mises distribution for two data sets of input values of constants.

3. EXAMPLE AND SOLVING

First Example

Using the MathCad program we are giving hereinafter, we can verify that the new function is a probability density.

$\mathtt{ORIGIN} \equiv 1$

$\mathtt{k1} := 2.3 \qquad \mathtt{k2} := 1$

$\mu 1 := 1 \qquad \mu 2 := 2$

$$f(x,y) := \frac{1}{4\cdot\pi^2\cdot I0(k1)\cdot I0(k2)}\cdot\exp\left(k1\cdot\cos\left(x-\mu 1\right)+k2\cdot\cos\left(y-\mu 2\right)\right)$$

In this case

$$\int_{\mu 1-\pi}^{\mu 1+\pi}\int_{\mu 2-\pi}^{\mu 2+\pi} f(x,y)\ dy\ dx = 1$$

Using the following program sequence, we obtained the graph of the 2-dimensional Mises distribution surface, given in Figure 1 and Figure 2 shows the contours of the surface.

$\mathtt{n1} := 16 \qquad \mathtt{n2} := 16$

$\mathtt{i1} := 1 .. \ \mathtt{n1}+1$

$\mathtt{i2} := 1 .. \ \mathtt{n2}+1$

$$xt_{i1} := \mu 1-\pi+\frac{i1-1}{n1}\cdot 2\cdot\pi$$

$$yt_{i2} := \mu 2 - \pi + \frac{i2-1}{n2}\cdot 2\cdot \pi$$

$$A_{i1,\,i2} := f\left(xt_{i1},\, yt_{i2}\right)$$

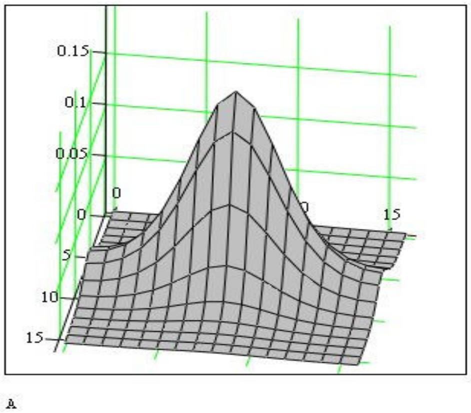

FIGURE 1. 2-dimensional Mises distribution surface in the first case

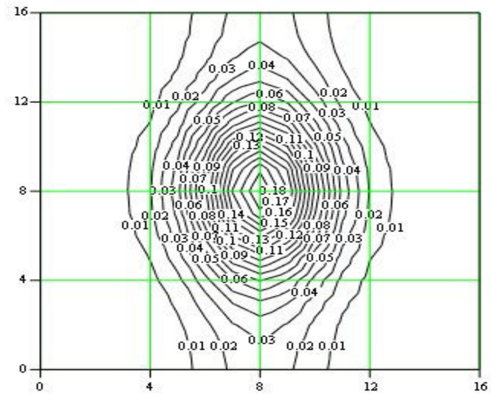

FIGURE 2. Contours of 2-dimensional Mises distribution surface in the first case

A part of the function f(x,y) values are presented below.

A =

	1	2	3	4	5	6
1	$2.60787\cdot 10^{-4}$	$2.81414\cdot 10^{-4}$	$3.49533\cdot 10^{-4}$	$4.83485\cdot 10^{-4}$	$7.08893\cdot 10^{-4}$	$1.03939\cdot 10^{-3}$
2	$3.10686\cdot 10^{-4}$	$3.35259\cdot 10^{-4}$	$4.16412\cdot 10^{-4}$	$5.75995\cdot 10^{-4}$	$8.44531\cdot 10^{-4}$	$1.23826\cdot 10^{-3}$
3	$5.11506\cdot 10^{-4}$	$5.51962\cdot 10^{-4}$	$6.85571\cdot 10^{-4}$	$9.48305\cdot 10^{-4}$	$1.39042\cdot 10^{-3}$	$2.03865\cdot 10^{-3}$
4	$1.07872\cdot 10^{-3}$	$1.16404\cdot 10^{-3}$	$1.44581\cdot 10^{-3}$	$1.99989\cdot 10^{-3}$	$2.93227\cdot 10^{-3}$	$4.29934\cdot 10^{-3}$
5	$2.60114\cdot 10^{-3}$	$2.80687\cdot 10^{-3}$	$3.48631\cdot 10^{-3}$	$4.82237\cdot 10^{-3}$	$7.07063\cdot 10^{-3}$	0.01037
6	$6.27216\cdot 10^{-3}$	$6.76824\cdot 10^{-3}$	$8.40658\cdot 10^{-3}$	0.01163	0.01705	0.025
7	0.01323	0.01427	0.01773	0.02452	0.03596	0.05272
8	0.02178	0.0235	0.02919	0.04037	0.0592	0.0868
9	0.02594	0.028	0.03477	0.0481	0.07052	0.1034
10	0.02178	0.0235	0.02919	0.04037	0.0592	0.0868
11	0.01323	0.01427	0.01773	0.02452	0.03596	0.05272
12	$6.27216\cdot 10^{-3}$	$6.76824\cdot 10^{-3}$	$8.40658\cdot 10^{-3}$	0.01163	0.01705	0.025
13	$2.60114\cdot 10^{-3}$	$2.80687\cdot 10^{-3}$	$3.48631\cdot 10^{-3}$	$4.82237\cdot 10^{-3}$	$7.07063\cdot 10^{-3}$	0.01037
14	$1.07872\cdot 10^{-3}$	$1.16404\cdot 10^{-3}$	$1.44581\cdot 10^{-3}$	$1.99989\cdot 10^{-3}$	$2.93227\cdot 10^{-3}$	$4.29934\cdot 10^{-3}$
15	$5.11506\cdot 10^{-4}$	$5.51962\cdot 10^{-4}$	$6.85571\cdot 10^{-4}$	$9.48305\cdot 10^{-4}$	$1.39042\cdot 10^{-3}$	$2.03865\cdot 10^{-3}$
16	$3.10686\cdot 10^{-4}$	$3.35259\cdot 10^{-4}$	$4.16412\cdot 10^{-4}$	$5.75995\cdot 10^{-4}$	$8.44531\cdot 10^{-4}$	$1.23826\cdot 10^{-3}$

Using the next MathCad sequence we obtained a complete set of function values, including all those values higher than 0.01.

$$B := \left|\begin{array}{l} B \leftarrow A \\ \text{for } i1 \in 1 .. \; n1+1 \\ \quad \text{for } i2 \in 1 .. \; n2+1 \\ \quad\quad B_{i1,\,i2} \leftarrow 0 \quad \text{if } B_{i1,\,i2} < 0.01 \\ B \end{array}\right.$$

We obtained the next values:

B =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0	0	0	0	0
6	0	0	0	0.01	0.02	0.02	0.03	0.04	0.05	0.04	0.03	0.02	0.02	0.01	0	0	0
7	0.01	0.01	0.02	0.02	0.04	0.05	0.07	0.09	0.1	0.09	0.07	0.05	0.04	0.02	0.02	0.01	0.01
8	0.02	0.02	0.03	0.04	0.06	0.09	0.12	0.15	0.16	0.15	0.12	0.09	0.06	0.04	0.03	0.02	0.02
9	0.03	0.03	0.03	0.05	0.07	0.1	0.14	0.18	0.19	0.18	0.14	0.1	0.07	0.05	0.03	0.03	0.03
10	0.02	0.02	0.03	0.04	0.06	0.09	0.12	0.15	0.16	0.15	0.12	0.09	0.06	0.04	0.03	0.02	0.02
11	0.01	0.01	0.02	0.02	0.04	0.05	0.07	0.09	0.1	0.09	0.07	0.05	0.04	0.02	0.02	0.01	0.01
12	0	0	0	0.01	0.02	0.02	0.03	0.04	0.05	0.04	0.03	0.02	0.02	0.01	0	0	0
13	0	0	0	0	0	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

One can notice that the considered von Mises distribution has its maxima

$$B_{\frac{n1}{2}+1,\,\frac{n2}{2}+1} = 0.1917034122$$

which has been calculated in one point given by relations

$$iex := \frac{n1}{2} + 1 \qquad iey := \frac{n2}{2} + 1 \qquad xte := \mu 1 - \pi + \frac{iex - 1}{n1} \cdot 2 \cdot \pi$$

$$yte := \mu 2 - \pi + \frac{iey - 1}{n2} \cdot 2 \cdot \pi$$

having the coordinates

$$xte = 1 \qquad yte = 2$$

Second Example

We considered the next input data

$$k1 := 1 \qquad k2 := 2.5$$

$$\mu 1 := 2 \qquad \mu 2 := 1$$

$$f(x, y) := \frac{1}{4 \cdot \pi^2 \cdot I0(k1) \cdot I0(k2)} \cdot \exp\left(k1 \cdot \cos(x - \mu 1) + k2 \cdot \cos(y - \mu 2)\right)$$

for that the value of the double integral considered bellow

$$\int_{\mu 1-\pi}^{\mu 1+\pi} \int_{\mu 2-\pi}^{\mu 2+\pi} f(x, y)\, dy\, dx = 1$$

shows that the function meets the condition (5) of being a probability distribution.

The graph of the 2-dimensional von Mises disribution corresponding to this data sets is depicted in Figure 3. In Figure 4 the corresponding contour lines are given.

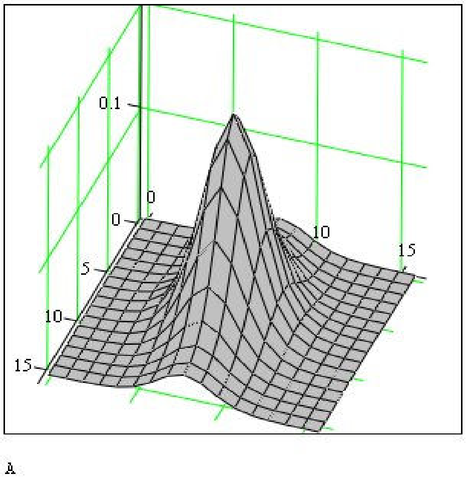

FIGURE 3. 2-dimensional Mises distribution surface in the second case

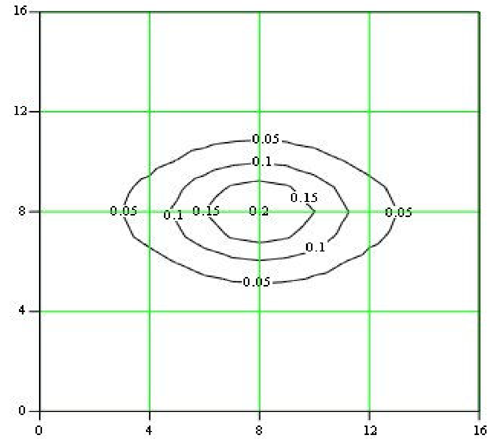

FIGURE 4. Contours of 2-dimensional Mises distribution surface in the second case

The set of function values obtained in this case is presented bellow.

B =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0
2	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0
3	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0
4	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0
5	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0
6	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0
7	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0
8	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0
9	0	0	0	0	0.02	0.04	0.1	0.17	0.2	0.17	0.1	0.04	0.02	0	0	0
10	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0
11	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0
12	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0
13	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0
14	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0
15	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0
16	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0

For this case the considered von Mises distribution has its maxima

$$B_{\frac{n1}{2}+1,\,\frac{n2}{2}+1} = 0.2013909646$$

at the coordinates

$$xte = 2 \qquad yte = 1$$

The considerations presented in the paper may lead to the future data modeling using the 2-dimensional von Mises distribution.

4. CONCLUSIONS

Analyzing the two cases presented in the paper one can notice that the distribution becomes very concentrated about the angle μ with k being a measure of the concentration. In fact, as k increases, the distribution approaches a normal distribution in x with mean μ and variance 1/k. The 2-dimensional von Mises distribution is a continuous probability distribution and it can be used in data modeling certain problems.

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