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CONSIDERATIONS UPON APPLYING SERIES EXPANSION TO THE VON MISES 2-DIMENSIONAL DISTRIBUTION

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ABSTRACT:
In this paper we present the opportunity to replace the 2-dimensional Mises distribution rule by its series expansion, analyzing two approximate expansions comparatively. Calculations have been done in MathCad.
KEYWORDS:
von Mises distribution, Series expansion, Data modeling

1. INTRODUCTION

In this paper we present the method to apply a series expansion to the 2-dimensional Mises distribution rule. We present in the Section 2 the theoretic laws that allow as to apply a series expansion to the 2-dimensional Mises distribution rule. Having the one-dimensional von Mises distribution rule, we introduced the partial functions that helped to construct the needed function. We present in comparison two truncated series that approximate the 2-dimensional Mises rule, sustained by calculations in MathCad. In Section 3 the results are exposed and Section 4 includes the conclusions of the paper.

2. METHODOLOGY

The von Mises distribution is a "natural" distribution for circular attributes (4), e.g. angles, time of day, day of the year, phase of the moon, etc..

The paper (1) presents the one-dimensional von Mises probability density function, having the form

$$f(x) = \frac{e^{k\cos(x-\mu)}}{2\pi I_0(k)} \tag{1}$$

where x belongs to a 2π length interval, $I_0(x)$ is the modified Bessel function of order 0, and k and μ are two real parameters, with k>0.

Presented paper denotes observations above the series expansion convergence for 2-dimensional von Mises distribution rule, expressed by

$$f(x, y) = \frac{e^{k_1\cos(x-\mu_1)}}{2\pi I_0(k_1)} \cdot \frac{e^{k_2\cos(x-\mu_2)}}{2\pi I_0(k_2)} \tag{2}$$

defined on $(x, y) \in [\mu_1 - \pi, \mu_1 + \pi] \times [\mu_2 - \pi, \mu_2 + \pi]$, the independent variables being x and y, $k_1>0$, $k_2>0$, μ_1 and μ_2 being real constants and $I_0(t)$ is the modified Bessel function of order 0.

The function (2) is a probability density because it is positive and its integral is equal to one (2, 3).

ORIGIN≡ 1

$k1 := 1 \qquad k2 := 2.5$

$\mu 1 := 2 \qquad \mu 2 := 1$

$$f(x,y) := \frac{1}{4\cdot\pi^2\cdot I0(k1)\cdot I0(k2)}\cdot\exp\left(k1\cdot\cos\left(x-\mu 1\right) + k2\cdot\cos\left(y-\mu 2\right)\right)$$

$$\int_{\mu 1-\pi}^{\mu 1+\pi}\int_{\mu 2-\pi}^{\mu 2+\pi} f(x,y)\,dy\,dx =$$

We consider

$np := 5$

and the Mises fuction has its series expansion

$$fx1(x) := \frac{1}{2\cdot\pi}\cdot\left[1+\frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos\left[j\cdot\left(x-\mu 1\right)\right]\right] \tag{3}$$

This function has the next graphic representation on domain $x \in \left[\mu_1 - \pi,\, \mu_1 + \pi\right]$

$n1 := 16$

$i := 1 .. n1 + 1$

$xt_i := \mu 1 - \pi + \frac{i-1}{n1}\cdot 2\cdot\pi$

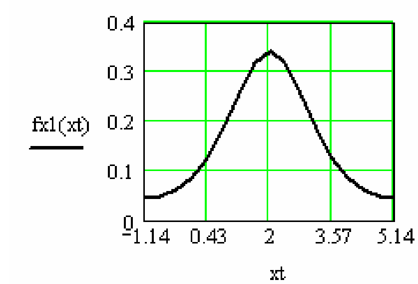


FIGURE 1. The graphic representation of the partial function fx1(x)

and it has a couple of values

$fx1(xt)^T =$

	1	2	3	4	5	6	7	8	9	10
1	0.046	0.05	0.062	0.086	0.126	0.184	0.255	0.317	0.342	0.317

respectively all the rounded values

$v := \text{augment}(xt, fx1(xt))$

$v^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	-1.1	-0.7	-0.4	0	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6	4	4.4	4.7	5.1
2	0	0	0.1	0.1	0.1	0.2	0.3	0.3	0.3	0.3	0.3	0.2	0.1	0.1	0.1	0	0

Let, for instance

$$fy1(y) := \frac{1}{2\cdot\pi}\cdot\left[1+\frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos\left[j\cdot\left(y-\mu 2\right)\right]\right] \tag{4}$$

be a function, whose graphic representation on domain $y \in [\mu_2 - \pi, \mu_2 + \pi]$ is

$$n2 := 16$$
$$j := 1 .. n2 + 1$$
$$yt_j := \mu 2 - \pi + \frac{j-1}{n2} \cdot 2 \cdot \pi$$

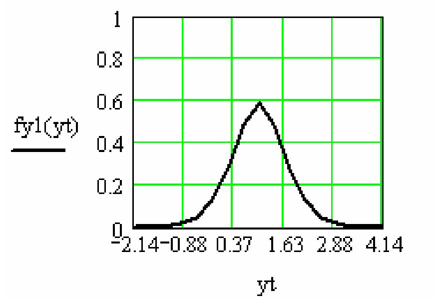


FIGURE 2. The graphic representation of the partial function fy1(y)

and its values are presented bellow

$$v := \mathrm{augment}(yt, fy1(yt))$$

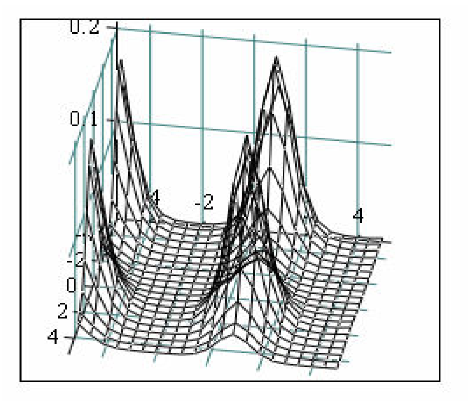
$v^T =$

	1	2	3	4	5	6	7	8
1	-2.14	-1.75	-1.36	-0.96	-0.57	-0.18	0.21	0.61
2	$3.43 \cdot 10^{-3}$	$5.17 \cdot 10^{-3}$	$8.32 \cdot 10^{-3}$	0.02	0.05	0.13	0.28	0.49

With the help of these two partial functions we construct the two variables probability density function

$$F1(x, y) := (fx1(x) \cdot fy1(y))$$

which has into an expanded domain the graphic representation depicted in Figure 3.



F1

FIGURE 3. The probability density function F1(x, y)

We introduce now the matrix

$$A_{i,j} := f(xt_i, yt_j)$$

Let

$$G(x,y) := \left[\frac{1}{2\cdot\pi}\cdot\left[1 + \frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos[j\cdot(x-\mu 1)]\right]\right]\cdot\left[\frac{1}{2\cdot\pi}\cdot\left[1 + \frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos[j\cdot(y-\mu 2)]\right]\right] \quad (5)$$

be defined on domain $(x, y) \in [\mu_1 - \pi, \mu_1 + \pi] \times [\mu_2 - \pi, \mu_2 + \pi]$. Having the notation

$$H_{i,j} := G(xt_i, yt_j)$$

we can obtain its graphic representation depicted in Figure 4. Figure 5 presents the contours for this surface.

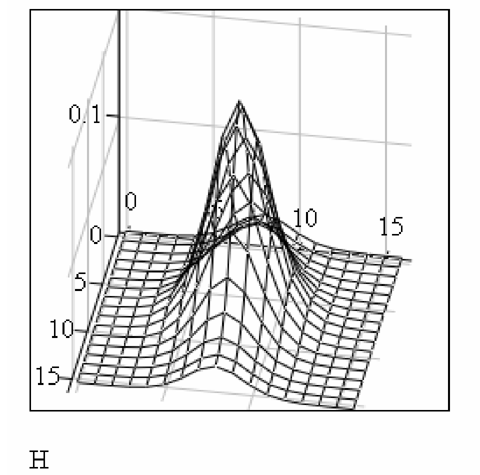


FIGURE 4. The graphic representation of G(x, y)

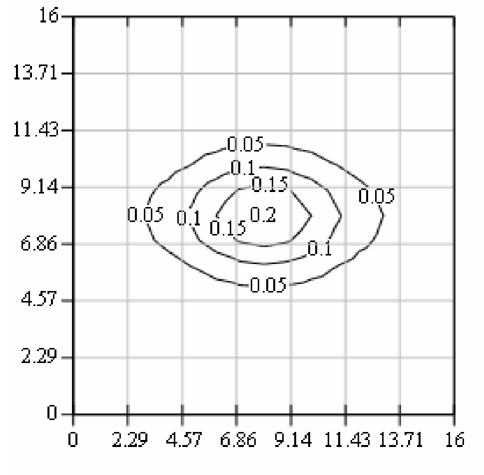


FIGURE 5. Contours of G(x, y) surface

Now we analyze the case that

$$np := 10$$

and

$$F2(x,y) := (fx2(x)\cdot fy2(y))$$

the expressions of fx2 and fy2 being the same like of their homologous, case in which np is replaced with 10.

Let

$$Q(x,y) := \frac{1}{2\cdot\pi}\cdot\left[1+\frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k1}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos[j\cdot(x-\mu1)]\right]\cdot\left[\frac{1}{2\cdot\pi}\cdot\left[1+\frac{2}{\displaystyle\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{0+2\cdot p}}{p!\cdot\Gamma(0+p+1)}}\cdot\sum_{j=1}^{np}\sum_{p=0}^{np}\frac{\left(\frac{k2}{2}\right)^{j+2\cdot p}}{p!\cdot\Gamma(j+p+1)}\cdot\cos[j\cdot(y-\mu2)]\right]\right] \tag{6}$$

be defined on domain $[\mu_1-\pi,\mu_1+\pi]\times[\mu_2-\pi,\mu_2+\pi]$. The representation of F2(x, y) is the same like of its homologous depicted in Figure 3 and the representation of Q(x, y) surface is the same like one depicted in Figure 4.

3. RESULTS AND INTERPRETATIONS

The exact probability density values can be partial viewed in the next table:

A =

	1	2	3	4	5	6	7
1	$1.836\cdot10^{-4}$	$2.221\cdot10^{-4}$	$3.819\cdot10^{-4}$	$8.595\cdot10^{-4}$	$2.237\cdot10^{-3}$	$5.824\cdot10^{-3}$	0.013
2	$1.982\cdot10^{-4}$	$2.397\cdot10^{-4}$	$4.121\cdot10^{-4}$	$9.274\cdot10^{-4}$	$2.414\cdot10^{-3}$	$6.284\cdot10^{-3}$	0.014
3	$2.461\cdot10^{-4}$	$2.977\cdot10^{-4}$	$5.119\cdot10^{-4}$	$1.152\cdot10^{-3}$	$2.999\cdot10^{-3}$	$7.806\cdot10^{-3}$	0.018
4	$3.405\cdot10^{-4}$	$4.118\cdot10^{-4}$	$7.081\cdot10^{-4}$	$1.593\cdot10^{-3}$	$4.148\cdot10^{-3}$	0.011	0.024
5	$4.992\cdot10^{-4}$	$6.038\cdot10^{-4}$	$1.038\cdot10^{-3}$	$2.336\cdot10^{-3}$	$6.081\cdot10^{-3}$	0.016	0.036
6	$7.319\cdot10^{-4}$	$8.854\cdot10^{-4}$	$1.522\cdot10^{-3}$	$3.425\cdot10^{-3}$	$8.917\cdot10^{-3}$	0.023	0.052
7	$1.012\cdot10^{-3}$	$1.225\cdot10^{-3}$	$2.106\cdot10^{-3}$	$4.738\cdot10^{-3}$	0.012	0.032	0.072
8	$1.258\cdot10^{-3}$	$1.521\cdot10^{-3}$	$2.615\cdot10^{-3}$	$5.885\cdot10^{-3}$	0.015	0.04	0.09
9	$1.357\cdot10^{-3}$	$1.641\cdot10^{-3}$	$2.822\cdot10^{-3}$	$6.351\cdot10^{-3}$	0.017	0.043	0.097
10	$1.258\cdot10^{-3}$	$1.521\cdot10^{-3}$	$2.615\cdot10^{-3}$	$5.885\cdot10^{-3}$	0.015	0.04	0.09
11	$1.012\cdot10^{-3}$	$1.225\cdot10^{-3}$	$2.106\cdot10^{-3}$	$4.738\cdot10^{-3}$	0.012	0.032	0.072
12	$7.319\cdot10^{-4}$	$8.854\cdot10^{-4}$	$1.522\cdot10^{-3}$	$3.425\cdot10^{-3}$	$8.917\cdot10^{-3}$	0.023	0.052
13	$4.992\cdot10^{-4}$	$6.038\cdot10^{-4}$	$1.038\cdot10^{-3}$	$2.336\cdot10^{-3}$	$6.081\cdot10^{-3}$	0.016	0.036
14	$3.405\cdot10^{-4}$	$4.118\cdot10^{-4}$	$7.081\cdot10^{-4}$	$1.593\cdot10^{-3}$	$4.148\cdot10^{-3}$	0.011	0.024
15	$2.461\cdot10^{-4}$	$2.977\cdot10^{-4}$	$5.119\cdot10^{-4}$	$1.152\cdot10^{-3}$	$2.999\cdot10^{-3}$	$7.806\cdot10^{-3}$	0.018
16	$1.982\cdot10^{-4}$	$2.397\cdot10^{-4}$	$4.121\cdot10^{-4}$	$9.274\cdot10^{-4}$	$2.414\cdot10^{-3}$	$6.284\cdot10^{-3}$	0.014

The approximate values that have been obtained are exposed in the next table:

Aap =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0	0
2	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0	0
3	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0	0
4	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0	0
5	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0	0
6	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0	0
7	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0	0
8	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0	0
9	0	0	0	0	0.02	0.04	0.1	0.17	0.2	0.17	0.1	0.04	0.02	0	0	0	0
10	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0	0
11	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0	0
12	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0	0
13	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0	0
14	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0	0
15	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0	0
16	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0	0
17	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0	0

The approximate series expansion, for np = 5, is shown bellow:

H =

	1	2	3	4	5	6	7
1	$1.585\cdot10^{-4}$	$2.389\cdot10^{-4}$	$3.848\cdot10^{-4}$	$8.372\cdot10^{-4}$	$2.266\cdot10^{-3}$	$5.805\cdot10^{-3}$	0.013
2	$1.71\cdot10^{-4}$	$2.579\cdot10^{-4}$	$4.153\cdot10^{-4}$	$9.036\cdot10^{-4}$	$2.445\cdot10^{-3}$	$6.265\cdot10^{-3}$	0.014
3	$2.124\cdot10^{-4}$	$3.203\cdot10^{-4}$	$5.158\cdot10^{-4}$	$1.122\cdot10^{-3}$	$3.037\cdot10^{-3}$	$7.781\cdot10^{-3}$	0.018
4	$2.938\cdot10^{-4}$	$4.43\cdot10^{-4}$	$7.134\cdot10^{-4}$	$1.552\cdot10^{-3}$	$4.201\cdot10^{-3}$	0.011	0.024
5	$4.309\cdot10^{-4}$	$6.495\cdot10^{-4}$	$1.046\cdot10^{-3}$	$2.276\cdot10^{-3}$	$6.16\cdot10^{-3}$	0.016	0.036
6	$6.317\cdot10^{-4}$	$9.523\cdot10^{-4}$	$1.534\cdot10^{-3}$	$3.337\cdot10^{-3}$	$9.031\cdot10^{-3}$	0.023	0.052
7	$8.738\cdot10^{-4}$	$1.317\cdot10^{-3}$	$2.121\cdot10^{-3}$	$4.616\cdot10^{-3}$	0.012	0.032	0.072
8	$1.085\cdot10^{-3}$	$1.636\cdot10^{-3}$	$2.635\cdot10^{-3}$	$5.733\cdot10^{-3}$	0.016	0.04	0.09
9	$1.171\cdot10^{-3}$	$1.765\cdot10^{-3}$	$2.843\cdot10^{-3}$	$6.187\cdot10^{-3}$	0.017	0.043	0.097
10	$1.085\cdot10^{-3}$	$1.636\cdot10^{-3}$	$2.635\cdot10^{-3}$	$5.733\cdot10^{-3}$	0.016	0.04	0.09
11	$8.738\cdot10^{-4}$	$1.317\cdot10^{-3}$	$2.121\cdot10^{-3}$	$4.616\cdot10^{-3}$	0.012	0.032	0.072
12	$6.317\cdot10^{-4}$	$9.523\cdot10^{-4}$	$1.534\cdot10^{-3}$	$3.337\cdot10^{-3}$	$9.031\cdot10^{-3}$	0.023	0.052
13	$4.309\cdot10^{-4}$	$6.495\cdot10^{-4}$	$1.046\cdot10^{-3}$	$2.276\cdot10^{-3}$	$6.16\cdot10^{-3}$	0.016	0.036
14	$2.938\cdot10^{-4}$	$4.43\cdot10^{-4}$	$7.134\cdot10^{-4}$	$1.552\cdot10^{-3}$	$4.201\cdot10^{-3}$	0.011	0.024
15	$2.124\cdot10^{-4}$	$3.203\cdot10^{-4}$	$5.158\cdot10^{-4}$	$1.122\cdot10^{-3}$	$3.037\cdot10^{-3}$	$7.781\cdot10^{-3}$	0.018
16	$1.71\cdot10^{-4}$	$2.579\cdot10^{-4}$	$4.153\cdot10^{-4}$	$9.036\cdot10^{-4}$	$2.445\cdot10^{-3}$	$6.265\cdot10^{-3}$	0.014

respectively, in gross:

Hap =

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0
2	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0
3	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0
4	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0
5	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0
6	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0
7	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0
8	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0
9	0	0	0	0	0.02	0.04	0.1	0.17	0.2	0.17	0.1	0.04	0.02	0	0	0
10	0	0	0	0	0.02	0.04	0.09	0.15	0.19	0.15	0.09	0.04	0.02	0	0	0
11	0	0	0	0	0.01	0.03	0.07	0.12	0.15	0.12	0.07	0.03	0.01	0	0	0
12	0	0	0	0	0	0.02	0.05	0.09	0.11	0.09	0.05	0.02	0	0	0	0
13	0	0	0	0	0	0.02	0.04	0.06	0.07	0.06	0.04	0.02	0	0	0	0
14	0	0	0	0	0	0.01	0.02	0.04	0.05	0.04	0.02	0.01	0	0	0	0
15	0	0	0	0	0	0	0.02	0.03	0.04	0.03	0.02	0	0	0	0	0
16	0	0	0	0	0	0	0.01	0.02	0.03	0.02	0.01	0	0	0	0	0

The committed error in this case is partial depicted in the next table:

$(A - H) \cdot 10^4 =$

	1	2	3	4	5	6	7	8	9	10
1	0.3	-0.2	-0	0.2	-0.3	0.2	0.1	-0.2	0.4	-0.2
2	0.3	-0.2	-0	0.2	-0.3	0.2	0	-0.3	0.4	-0.3
3	0.3	-0.2	-0	0.3	-0.4	0.2	0.1	-0.4	0.5	-0.4
4	0.5	-0.3	-0.1	0.4	-0.5	0.3	0.1	-0.5	0.7	-0.5
5	0.7	-0.5	-0.1	0.6	-0.8	0.5	0.1	-0.8	0.9	-0.8
6	1	-0.7	-0.1	0.9	-1.1	0.7	0.2	-1	1.4	-1
7	1.4	-0.9	-0.2	1.2	-1.6	1	0.2	-1.5	2	-1.5
8	1.7	-1.2	-0.2	1.5	-2	1.2	0.3	-1.8	2.4	-1.8
9	1.9	-1.2	-0.2	1.6	-2.1	1.3	0.3	-1.9	2.7	-1.9
10	1.7	-1.2	-0.2	1.5	-2	1.2	0.3	-1.8	2.4	-1.8
11	1.4	-0.9	-0.2	1.2	-1.6	1	0.2	-1.5	2	-1.5
12	1	-0.7	-0.1	0.9	-1.1	0.7	0.2	-1	1.4	-1
13	0.7	-0.5	-0.1	0.6	-0.8	0.5	0.1	-0.8	0.9	-0.8
14	0.5	-0.3	-0.1	0.4	-0.5	0.3	0.1	-0.5	0.7	-0.5
15	0.3	-0.2	-0	0.3	-0.4	0.2	0.1	-0.4	0.5	-0.4
16	0.3	-0.2	-0	0.2	-0.3	0.2	0	-0.3	0.4	-0.3

The same considerations in the case of np = 10, led to the next data:

$(A - W) \cdot 10^9 =$

	1	2	3	4	5	6	7	8	9
1	-1.35	0.56	0.91	-1.36	0.15	1.36	-1.21	-0.56	1.65
2	-1.45	0.61	0.98	-1.47	0.16	1.47	-1.31	-0.61	1.79
3	-1.8	0.76	1.22	-1.82	0.2	1.82	-1.63	-0.75	2.22
4	-2.49	1.05	1.68	-2.52	0.28	2.52	-2.25	-1.04	3.07
5	-3.66	1.53	2.47	-3.7	0.41	3.69	-3.3	-1.53	4.5
6	-5.36	2.25	3.62	-5.42	0.61	5.41	-4.84	-2.24	6.6
7	-7.42	3.11	5	-7.5	0.84	7.49	-6.69	-3.1	9.12
8	-9.21	3.86	6.21	-9.31	1.04	9.3	-8.31	-3.85	11.33
9	-9.94	4.17	6.71	-10.05	1.12	10.04	-8.97	-4.15	12.23
10	-9.21	3.86	6.21	-9.31	1.04	9.3	-8.31	-3.85	11.33
11	-7.42	3.11	5	-7.5	0.84	7.49	-6.69	-3.1	9.12
12	-5.36	2.25	3.62	-5.42	0.61	5.41	-4.84	-2.24	6.6
13	-3.66	1.53	2.47	-3.7	0.41	3.69	-3.3	-1.53	4.5
14	-2.49	1.05	1.68	-2.52	0.28	2.52	-2.25	-1.04	3.07
15	-1.8	0.76	1.22	-1.82	0.2	1.82	-1.63	-0.75	2.22
16	-1.45	0.61	0.98	-1.47	0.16	1.47	-1.31	-0.61	1.79
17	-1.35	0.56	0.91	-1.36	0.15	1.36	-1.21	-0.56	1.65

By further calculating the difference between the probability density values and the approximate series expansion values, in the presented two cases, we get the results:

$$\max(A - H) = \blacksquare \qquad \max(A - W) = \blacksquare$$

This results conclude that the second modeling is better than first.

4. CONCLUSIONS

The presented evaluations permit a future analyze of all existent results which have been obtained until now (5), and may lead to the future study of the series expansion method applied to other probability density rules and, as well, to some two variable functions.

The considerations presented in the paper may also lead to data modeling using the 2-dimensional Mises distribution.

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