JOURNAL OF ENGINEERING
annals OF
FACULTY OF ENGINEERING HUNEDOARA
Tome V (year 2007), Fascicole 3, (ISSN 1584 – 2673)

# EXTENDING THE POLAR CUT BASED SET INTERPOLATION AND REVISION METHODS TO THE CASE OF POLYGONAL AND GAUSSIAN SHAPED FUZZY SETS

Zsolt Csaba JOHANYÁK, Rafael Pedro ALVAREZ GIL, Szilveszter KOVÁCS

KECSKEMÉT COLLEGE, GAMF FACULTY,
INSTITUTE OF INFORMATION TECHNOLOGIES, HUNGARY

**ABSTRACT**:
Fuzzy systems applying a fuzzy rule interpolation based inference technique offer an acceptable output even in cases when none of the rule antecedents intersects or overlaps the input values. The polar cut based FEAT-p and SURE-p techniques present several advantages, being able to handle even such cases when the fuzzy sets of the system have different height values.
This paper extends the application area of these methods by introducing the polar cut calculation techniques for polygonal and Gaussian shaped fuzzy sets.
**KEYWORDS**:
polar cut, polygonal membership function, Gaussian membership function

## 1. INTRODUCTION

Fuzzy systems with multiple input state variables require a very high number of rules in order to ensure a full coverage of the input space. However, while the classical fuzzy inference methods (e.g. (15), (18), etc.) require the dense character of the rule base, the reasoning methods based on fuzzy rule interpolation can ensure an acceptable output even in cases when there is no rule whose antecedent part would intersect or overlap the observation.

The development and research related to fuzzy rule interpolation was started by Kóczy and Hirota, who proposed the first method called linear fuzzy interpolation or KH method (13), which was able to calculate the conclusion taking into consideration two rules whose antecedent part flanked the observation. Recognising its limitations several enhancements of the KH technique as well as methods trying new approaches has been proposed. They can be divided into two groups depending on whether they calculate the conclusion directly or an auxiliary rule is produced first. The members of both groups can be applied to fuzzy logic control (16).

The most relevant members of the first group are the KH method (13); the MACI (19) (Tikk and Baranyi), which avoids the abnormal conclusion by applying a co-ordinate transformation; the FIVE (12) (Kovács and Kóczy), which solves the problem of rule interpolation in the so called vague environment; the interpolative reasoning based on graduality introduced by Bouchon-Meunier, Marsala and Rifqi; the CRF

(14) (Kóczy, Hirota and Gedeon), which is based on the conservation of the relative fuzziness; the IMUL (21) (Wong, Gedeon and Tikk), which uses the co-ordinate transformation and the concept of relative fuzziness in case of multidimensional input spaces; and the method proposed by Dubois and Prade (DP) (5).

The methods belonging to the second group follow the concepts of the generalized methodology of fuzzy rule interpolation (GM) (2) introduced by Baranyi, Kóczy and Gedeon. Relevant members of this group are the techniques suggested by Branyi et al in (1)(2)(3); the method IGRV (6) (Huang és Shen) based on the generalized representative values; the LESFRI (8) (Johanyák and Kovács), which is based on the method of least squares; the FRIPOC (7) (Johanyák and Kovács) using the concept of linguistic term shifting and polar cuts (figure 1.); and the VEIN (10), which uses the vague environment for set interpolation and single rule reasoning.

The main advantages of the method FRIPOC (7), which is built up from the set interpolation technique FEAT-p (9) and the revision method SURE-p (7), are that

- it calculates the conclusion even in cases when one or more sets are subnormal, i.e. their height is smaller than 1;
- it calculates the conclusion even in cases when one cannot find at least two rules whose antecedent part surround the observation in each antecedent dimension (extrapolation capability);
- it is applicable in cases when the antecedent space of the fuzzy system is multidimensional.

The original versions of FEAT-p and SURE-p were developed for the widely used singleton, triangular and trapezoidal shaped fuzzy sets. This paper extends the applicability of the original methods by presenting the handling mode (polar cut calculation) of the polygonal shaped and Gaussian fuzzy sets.

The rest of this paper is organized as follows. Section 2 recalls the basic ideas of the fuzzy rule interpolation method FRIPOC. Section 3 introduces the polar cut calculation mode for polygonal and Gaussian membership functions. The conclusions are presented in section 4.

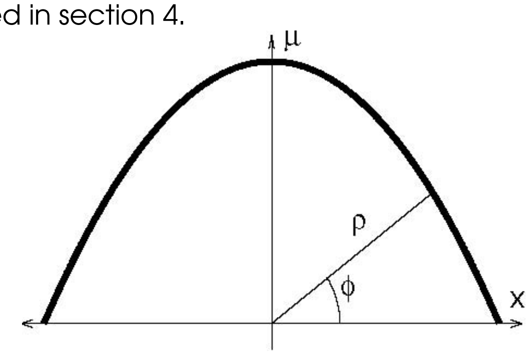


FIGURE 1. POLAR CUT

## 2. THE FRIPOC METHOD

The Fuzzy Rule Interpolation based on POlar Cuts (7) solves the task of fuzzy reasoning in two steps conform to the GM (2). First a new rule is interpolated whose antecedent part is in the same position as the observation in each antecedent dimension. The expression "same position" means that in each partition the reference point of the observation and the reference point of the rule antecedent set are identical. FRIPOC uses the centre of the core as reference point.

The new rule is determined in three stages. First the shapes of the antecedent sets are calculated using the set interpolation technique FEAT-p (9) separately in each antecedent dimension. Its main idea is that all sets of the partition are shifted horizontally into the interpolation point (reference point of the observation), i.e. their reference points will be identical with the interpolation point. Next the shape of the new set is calculated by its polar cuts. For each polar level the polar distance is determined as a weighted average of the corresponding polar distances of the overlapped known sets. The position of the consequent sets is calculated in the second stage using an adapted version of the Shepard interpolation (17). Next (stage 3) one calculates the shape of the consequent sets by FEAT-p in an identical way as seen in case of the antecedent sets (stage 1).

The second step of FRIPOC determines the conclusion from the observation and the previously generated auxiliary rule using the method SURE-p (7). The Single rUle Reasoning based on polar cuts calculates the differences between the polar distances corresponding to the observation and the antecedent of the interpolated rule in each antecedent dimension and for each polar level. Next an average difference is determined for each polar level. One calculates the conclusion by modifying the consequent of the interpolated rule by the average differences followed by a control and correction algorithm in order to ensure the validity of the new fuzzy set.

## 3. POLAR CUT CALCULATION FOR POLYGONAL AND GAUSSIAN MEMBERSHIP FUNCTIONS

During polar cut calculation one aims the determination of the polar distance of a point knowing the polar angle ($\varphi$). The point is situated on the curve describing the shape (membership function - $\mu$) of a fuzzy set (*A*), thus one seeks a function in form of

$$\rho = f(\mu_A, \varphi). \tag{1}$$

In order to determine the desired value two co-ordinate transformations are needed. First one translates horizontally the co-ordinate system into the reference point of the fuzzy set (*RP*), which means a transformation between the original Cartesian co-ordinates (*x*,*y*) and the new Cartesian co-ordinates (*x′*,*y′*). Next a second transformation is done between the co-ordinates (*x,′y′*) and the polar co-ordinates (*ρ*,*φ*). Figure 2 shows the three co-ordinate systems.

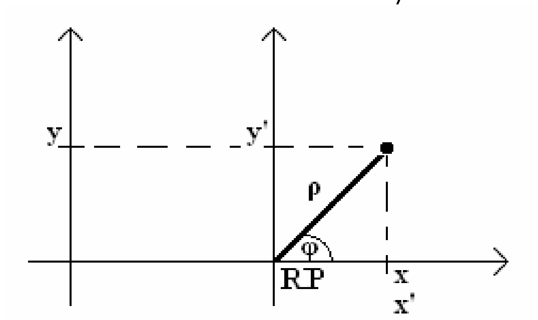

FIGURE 2. CO-ORDINATE SYSTEMS

The relationship between the co-ordinate systems is described by the following equations

$$x' = x - RP\ , \tag{2}$$

$$y' = y\ , \tag{3}$$

$$x' = \rho \cos\varphi\ , \tag{4}$$

$$y' = \rho \sin \varphi \,. \tag{5}$$

The steps of the determination of the polar distance are not the same in case of different set shapes, therefore they are presented separately in the following two sections.

### 3.1. POLYGONAL SHAPED FUZZY SETS

In course of the discussion of the polar cut calculation we consider the polygonal shape as a generalization of the shape types singleton, triangle and trapezoidal, i.e. the presented calculation mode is applicable for their case as well. In practical applications in case of polygonal shaped fuzzy sets the membership function is defined by the vertices of the shape

$$\mu_A = y_0 / x_0 + y_1 / x_1 + ... + y_{n-1} / x_{n-1} \,, \tag{6}$$

where $x_i$,$y_i$ are the Cartesian co-ordinates of the *i*th vertex, *n* is the number of the vertices and $x_0 \le x_1 \le \ldots \le x_{n-1}, n \ge 1$ holds.

First one has to calculate the height of the fuzzy set (7) followed by the determination of the endpoints of the core (8) (9)

$$y_{\max} = height(A) = \max\{ y_0, y_1, \ldots, y_{n-1} \} \,, \tag{7}$$

$$x_{\max 1} = \min\{ x_i \mid y_i = y_{\max} \} \,, \tag{8}$$

$$x_{\max 2} = \max\{ x_i \mid y_i = y_{\max} \} \,. \tag{9}$$

Considering the centre of the core as reference point (*RP*) its position is given by

$$RP = (x_{\max 1} + x_{\max 2}) / 2 \,. \tag{10}$$

Supposing a convex fuzzy set, one can assume that $y_0 \le y_1 \le \ldots \le y_{i-1} \le y_i = y_{max} \ge y_{i+1} \ge \ldots \ge y_{n-1}$ for some not necessarily unique $i$ ($0 \le i \le n-1$). Furthermore $y_0 = 0$, $y_{n-1} = 0$ and $y_i \ne 0$ for $0 < i < n-1$. The only exception is the case of the singleton shaped fuzzy set, where only one ($x_0$,$y_0$) point is given and $y_0 > 0$.

Next one determines the $\Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\}$ polar angles corresponding to the vertices by

$$\varphi_i = \begin{cases} \arctan(y_i / (x_i - RP)) & if\ x_i > RP \\ \pi / 2 & if\ x_i = RP \quad (0 \le i \le n-1) \\ \arctan(y_i / (x_i - RP)) + \pi & if\ x_i < RP. \end{cases} \tag{11}$$

The edges are defined by the parameters $M = \{m_0, m_1, \ldots, m_{n-2}\}$ and $B = \{b_0, b_1, \ldots, b_{n-2}\}$ as follows

$$\begin{cases} y = m_i x_i + b_i & where\ m_i = (y_{i+1} - y_i)/(x_{i+1} - x_i)\ and\ b_i = y_i - m_i x_i,\ if\ x_i \ne x_{i+1} \\ x = x_i & if\ x_i = x_{i+1}\ (vertical\ edge, m_i\ and\ b_i\ are\ undefined) \end{cases} \tag{12}$$

The calculations of the value of $\rho$ for an arbitrary value of $\varphi$ are started from

$$y = mx + b \,. \tag{13}$$

Next, using the equations (2) and (3) after the substitutions one gets

$$y' = m(x' + RP) + b \,. \tag{14}$$

Then using (4) and (5) one gets

$$\rho \sin\varphi = m(\rho\cos\varphi + RP) + b \,. \tag{15}$$

Expressing $\rho$ results

$$\rho = (mRP + b)/(\sin\varphi - m\cos\varphi) \,. \tag{16}$$

Considering some special cases related to the polar angle and the shape type, the final formula of $\rho$ for an arbitrary value of $\varphi$ ($\varphi_0 \le \varphi \le \varphi_{n-1}$) is

$$\rho = \begin{cases} y_{\max} & if \varphi = \pi/2 \\ (x_i - RP)/\cos\varphi_i & if \varphi = \varphi_i (0 \le i \le n-1) \; and \; \varphi \ne \pi/2 \\ (m_i RP + b_i)/(\sin\varphi - m_i\cos\varphi) & \begin{array}{l} if \varphi_i > \varphi > \varphi_{i+1} (0 \le i \le n-2), \varphi \ne \pi/2 \; and \\ x_i < x_{i+1} \; (m_i \; and \; b_i \; are \; undefined) \end{array} \\ (x_i - RP)/\cos\varphi & \begin{array}{l} if \varphi_i > \varphi > \varphi_{i+1} (0 \le i \le n-2), \varphi \ne \pi/2 \; and \\ x_i = x_{i+1} \; (m_i \; and \; b_i \; are \; undefined) \end{array} \\ 0 & otherwise. \end{cases} \tag{17}$$

It should be remarked that the fifth case ("0 - otherwise") occurs only by singleton shaped fuzzy sets. In case of that shape type the second, third and fourth cases are not applicable, because $\rho$ is *0* for every angle except for $\pi/2$, where $\rho = y_{max}$.

### 3.2. FUZZY SETS WITH GAUSSIAN MEMBERSHIP FUNCTIONS

Similar to other fuzzy rule interpolation methods FRIPOC requires a bounded support for the membership functions of the fuzzy sets. Denoting by $x_1$ and $x_2$ the lower respective upper endpoints of the partition and allowing only sets whose reference points are inside the interval $(x_1, x_2)$ ($x_1 \le RP \le x_2$) the resulting set shape is delimited by two vertical lines and a curve defined by

$$y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-RP)^2}{2\sigma^2}} \,, \tag{18}$$

where $\sigma$ is a parameter of the curve (figure 3).

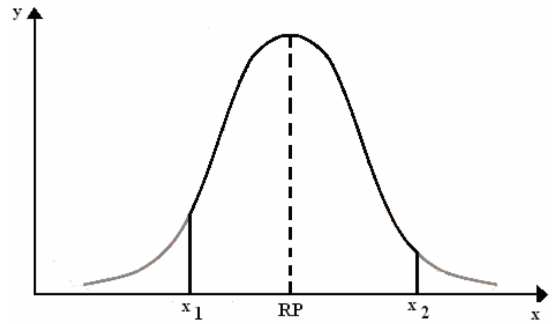


FIGURE 3. GAUSS CURVE

Considering only the Gaussian curve, the polar distance can be expressed from (18) after the substitutions defined by (2), (3), (4) and (5). The resulting formula is

$$\rho = \frac{1}{\sqrt{2\pi}\sigma \sin\varphi} e^{-\frac{\rho^2 \cos^2\varphi}{2\sigma^2}}. \tag{19}$$

Next one applies the constraints to the endpoints of the partition. The ordinate values that correspond to the break-points defined by the bounds of the partition are

$$y_1 = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_1 - RP)^2}{2\sigma^2}}, \tag{20}$$

$$y_2 = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_2 - RP)^2}{2\sigma^2}}. \tag{21}$$

The polar angles corresponding to the break-points are calculated by

$$\varphi_1 = \begin{cases} \pi/2 & if\ x_1 = RP \\ \arctan(y_1/(x_1 - RP)) + \pi & if\ x_1 < RP, \end{cases} \tag{22}$$

$$\varphi_2 = \begin{cases} \arctan\left(y_2/\left(x_2 - RP\right)\right) & if\ x_2 > RP \\ \pi/2 & if\ x_2 = RP. \end{cases} \tag{23}$$

Now one can calculate the value of the polar distance $\rho$ for an arbitrary value of the polar angle $\varphi$ ($0 \le \varphi \le \pi$) by

$$\rho = \begin{cases} (x_1 - RP)/\cos\varphi & if\ \varphi \ge \varphi_1 \\ \frac{1}{\sqrt{2\pi}\sigma \sin(\pi - \varphi)} e^{-\frac{\rho^2 \cos^2(\pi-\varphi)}{2\sigma^2}} & if\ \varphi_1 > \varphi > \pi/2 \\ \frac{1}{\sqrt{2\pi}\sigma \sin\varphi} e^{-\frac{\rho^2 \cos^2\varphi}{2\sigma^2}} & if\ \pi/2 \ge \varphi > \varphi_2 \\ (x_2 - RP)/\cos\varphi & if\ \varphi_2 \ge \varphi \end{cases} \tag{24}$$

The solution is simple in the first and the last cases. However, when the polar distance is defined by the Gaussian curve ($\varphi_2 < \varphi < \varphi_1$) the equation contains ρ in both of its sides with the exception of $\varphi = \pi / 2$, when $\rho = 1/\left(\sqrt{2\pi}\sigma\right)$. In the Gaussian case a numerical root finding method (e.g. bisection method (23), Newton-Raphson method (24), etc.) has to be applied. Despite its well known slow convergence, we used the bisection method in course of the implementation due to its simplicity and comprehensibility. It is easy to see that one can start iterating from the interval $[0, 1/\sqrt{2\pi}\sigma \sin\theta]$, where θ = π - φ in the second case and θ = φ in the third case in (24).

In case of bisection method the number of iterations (*n*) required to obtain a value of ρ with *k* exact decimal places is given by the formula

$$n = \log_2 \frac{10^k}{\sqrt{2\pi}\sigma \sin\theta} + 1. \qquad (25)$$

## 4. CONCLUSIONS

Despite its advantageous properties the wide application of the polar cut based FEAT-p and SURE-p techniques was delimited to the singleton, triangular and trapezoidal membership function types. This paper enlarged the possible practical application area of these methods by adding a special (Gaussian) and a general shape type to the pool of applicable membership functions. The presented calculation method has been integrated into the fuzzy rule interpolation Matlab ToolBox (11) and is available for download at (22). Future research will be focused on increasing the area of applications (20).

## REFERENCES

(1) BARANYI, P., KÓCZY, L. T.: A General and Specialised Solid Cutting Method for Fuzzy Rule Interpolation, in Journal BUSEFAL, URA-CNRS, Vol. 66., Toulouse, France, 1996, pp. 13-22.

[2] BARANYI, P., KÓCZY, L. T. and GEDEON, T. D.: A Generalized Concept for Fuzzy Rule Interpolation, in IEEE Transaction on Fuzzy Systems, ISSN 1063-6706, Vol. 12, No. 6, 2004. pp 820-837.

[3] BARANYI, P., MIZIK, S., KÓCZY, L.T., GEDEON, T. and NAGY, I.: Fuzzy Rule Base Interpolation Based on Semantic Revision, in Proceedings of the IEEE International Conference on System Man and Cybernetics (IEEE SMC'98), San Diego, USA, 1998, pp.1306-1311.

[4] BOUCHON-MEUNIER, B., MARSALA, C. and RIFQI, M.: Interpolative reasoning based on graduality, in Proceedings of the FUZZ-IEEE'2000, San Antonio, USA, 2000, pp. 483-487.

[5] DUBOIS, D. and PRADE, H.: Gradual inference rules in approximate reasoning, in Information Sciences, Vol. 61, 1992, pp. 103-122.

[6] HUANG, Z. H. and SHEN, Q.: Fuzzy interpolation with generalized representative values, in Proceedings of the UK Workshop on Computational Intelligence, 2004, pp. 161-171.

[7] JOHANYÁK, ZS. Cs. and KOVÁCS, SZ.: Fuzzy Rule Interpolation Based on Polar Cuts, in Computational Intelligence, Theory and Applications, Springer Berlin Heidelberg, 2006, pp. 499-511.

[8] JOHANYÁK, ZS. CS., KOVÁCS, Sz.: Fuzzy Rule Interpolation by the Least Squares Method, in Proceedings of the 7th International Symposium of Hungarian Researchers on Computational Intelligence (HUCI 2006), Budapest, Hungary, 2006, pp. 495-506.

[9] JOHANYÁK, ZS. CS., KOVÁCS, Sz.: Fuzzy set approximation using polar co-ordinates and linguistic term shifting, 4rd Slovakian-Hungarian Joint Symposium on Applied Machine Intelligence (SAMI 2006), , Herl'any, Slovakia, 2006, pp. 219-227.

[10] JOHANYÁK, Zs. Cs., KOVÁCS, Sz.: Vague Environment-based Two-step Fuzzy Rule Interpolation Method, in Proceedings of the 5th Slovakian-Hungarian Joint Symposium on Applied Machine Intelligence and Informatics (SAMI 2007), Poprad, Slovakia, 2007, pp. 189-200.
[11] JOHANYÁK, Zs. Cs., TIKK, D., KOVÁCS, Sz. and Wong, K. W: Fuzzy rule interpolation Matlab toolbox – FRI toolbox, in Proceedings of the IEEE World Congress on Computational Intelligence (WCCI'06), 15th Int. Conf. on Fuzzy Systems (FUZZ-IEEE'06), 2006, Vancouver, BC, Canada, pp. 1427-1433.
[12] KOVÁCS, Sz. and KÓCZY, L. T.: Application of an approximate fuzzy logic controller in an AGV steering system, path tracking and collision avoidance strategy, Fuzzy Set Theory and Applications, in Tatra Mountains Mathematical Publications, Mathematical Institute of Slovak Academy of Sciences, Vol. 16, Bratislava, Slovakia, 1999, pp. 456-467.
[13] KÓCZY, L. T. and HIROTA, K.: Approximate reasoning by linear rule interpolation and general approximation, in International Journal of Approximative Reasoning, Vol. 9, 1993, pp. 197–225.
[14] KÓCZY, L.T., HIROTA, K. and GEDEON, T. D.: Fuzzy rule interpolation by the conservation of relative fuzziness, in Journal of Advanced Computational Intelligence, Vol. 4/1, 2000, pp. 95-101.
[15] MAMDANI, E. H. and ASSILIAN, S.: An experiment in linguistic synthesis with a fuzzy logic controller, in International Journal of Man Machine Studies, Vol. 7, 1975, pp. 1-13.
[16] PRECUP, R. E., TOMESCU, M. L., PREITL, St., TAR, J. K. and PAUL, A. S.: Stability Analysis Approach for Fuzzy Logic Control Systems with Mamdani Type Fuzzy Logic Controllers, Control Engineering and Applied Informatics, Romanian Society of Control Engineering and Technical Informatics, Bucuresti, ISSN 1454-8658, 2007, Vol. 9, No. 1, pp. 3 - 10.
[17] SHEPARD, D.: A two dimensional interpolation function for irregularly spaced data, in Proceedings of the 23rd ACM International Conference, New York, USA, 1968, pp. 517-524.
[18] TAKAGI, T. and SUGENO, M.: Fuzzy identification of systems and its applications to modeling and control, in IEEE Transactions on System, Man and Cybernetics, Vol. 15, 1985, pp. 116-132.
[19] TIKK, D. and BARANYI, P.: Comprehensive analysis of a new fuzzy rule interpolation method, in IEEE Transactions on Fuzzy Systems, Vol. 8, 2000, pp. 281-296.
[20] TOMESCU, M. L., PREITL, ST, PRECUP, R. E., and TAR J. K.: Stability Analysis Method for Fuzzy Control Systems Dedicated Controlling Nonlinear Processes, Acta Polytechnica Hungarica, Vol. 4, No. 3, 2007, pag. 127-141.
[21] WONG, K. W., GEDEON, T. D. and TIKK, D.: An improved multidimensional α-cut based fuzzy interpolation technique, In Proc. Int. Conf. Artificial Intelligence in Science and Technology (AISAT’2000) , Hobart, Australia, 2000, pp. 29–32.
[22] http://fri.gamf.hu, website dedicated to fuzzy rule interpolation.
[23] http://mathworld.wolfram.com/Bisection.html
[24] http://mathworld.wolfram.com/NewtonsMethod.html