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THE MINIMALIST OF THE AXIOM SYSTEM OF THE BARBILIAN AFFINE STRUCTURE

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Abstract:
The system of the axioms of the affine Barbilian structure is minimal, this consists in building for each axiom (Ai) i=1...5 a model Mi in which the axiom Ai is not verified, but the other axioms are verified.

Keywords:
Minimalist, Barbilian affine structure, neighboured points, crossed lines.

1. INTRODUCTION

In this article are studied geometries within the uniqueness of the straight lines is not assured by the two distinct points. Two points through which two distinct straight lines can be assigned are called neighboured points.

Building an affine geometry upon the corpus $(Z_3, \oplus, \otimes)$ where $\oplus, \otimes$ are the addition and the multiplying modulo 3, whichever the two straight lines belonging to this geometry would be, their intersection will contain at least one point.

Building an affine geometry upon the corpus $(Z_4, \oplus, \otimes)$ where $\oplus, \otimes$ are the addition and the multiplying modulo 4, we will be able to build the points : $P=\{(x,y)/x,y \in Z_4\}=\{(0,0),(0,1),\ldots,(0,3),(1,0),(1,1),\ldots,(1,3),(3,0),\ldots,(3,3)\}$, and the straight lines can be defined such this: $D=\{(x_0, y_0) \oplus \lambda(x,y) \in Z_4, x_0, y_0, x, y, \lambda \in Z_4\}$. Within this geometry we can build two straight lines which will have at least two common points.

It is given $d_1=(2,1)+<(3,2)>=\{(2,1)+\lambda(3,2)/\lambda \in Z_4\}=\{(2,1),(1,3),(0,1),(3,3)\}$ and the straight line $d_2=(1,1)+<(1,0)>=\{(1,1)+\lambda(1,0)/\lambda \in Z_4\}= =\{(1,1),(2,1),(3,1),(0,1)\}$.

It can be noticed $\{(2,1),(0,1)\} \in d_2$, but this points are situated on the straight line d_1, too.

We will consider an abstract set P={…,p,q,…} whose elements I we will note points and an empty set D={…,H,G,…}, whose elements we will call straight lines. On the set PxP we will define a binary relation which is unreflexiv and symmetrical, „neighboured" and on the group DxD an equivalent parallel relation.

Definition 1: Two points' p and q are in neighboured relation if there are two distinct lines G and H in that way that p and q will be incident with G, but p and q are incident with H, too, and we will note it poq. In opposite case p is non-neighboured to q and we will note it $p \phi q$.

Definition 2: Two distinct lines G and H are parallel if there is not a point p in that way that p is incident with both G and H and we will note it like this $G\,||\,H$.

Definition 3: A point p is called the intersection point of the lines G and H if it belongs to the intersection of the sets G and H.

Definition 4: We consider that the line G cuts the line H, if the intersection of the sets G and H is empty.

Definition 5: The line G is crossed considering straight line H, and we can note it G©H, if any straight line G, noted G′ cuts any parallel line to H, noted H′ in a one single point.

Definition 6: We call $(P,D,o,||)$ a Barbilian structure if the following

Axioms are satisfied:

(A1) Through two non-neighboured points elapse unique straight-line

(A2) For any point p of a line G, there is a point $q\in G$ which is not neighboured with it.

(A3) (The axiom of the neighboured parallels)

There is for any pair $(p,G)\in P \times D$ a unique line H, noted $(p\,||\,G)$, in that way that $p\in H$ and $H\,||\,G$.

(A4) Any straight line is crossed at least once by another straight line.

(A5) If G and H are crossed lines and $p\in H$ is not neighboured to the intersection $G\cap H$ and if p is not neighboured to any points belonging to G, then any straight line through p, which crosses G, is crossed considering G.

Definition 7: An axiom system of an axiomatically theory T is called minimal, if each of its primary notions and relations is independent towards the others and if each of its axioms is independent from the others.

Let T be a semi-formalized axiomatic theory. The axiom A will be independent from the axiom systems A1, A2,..., An, if and only if there is an interpretation of the theory T, within which the axioms A1,A2,…,An are verified, but axiom A shall not take place. The interpretation obtained in this way is a model for the theory deducted from $\overline{A}$, A1,A2,…,An.

Definition 8: A set of axioms A1,A2,…,An, is called relative independent if for any i=1,...,n there is a model which will verify the axioms A1,A2,…,Ai-1 and it is not verified the axiom Ai.

In the followings we will demonstrate the minimalist of the affine Barbilian structure.

2. A MODEL OF BARBILIAN AFFINE STRUCTURE

We will demonstrate that the semi-formalized theory deducted from the five axioms which has as primary notions the points P, the straight lines D and the relations of parallelism and non-neighbouring is consistent.

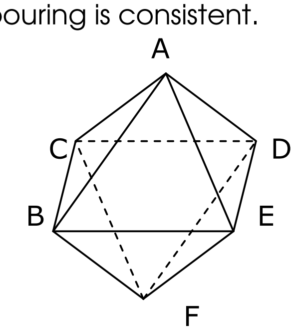

FIGURE 1. Model of Barbilian affine structure

We will offer then a model of this theory. Let it be the polyhedron in Euclidian space with its apex A,B,C,D,E,F, as in figure 1.

The set of the points P ={{A,B,C},{A,D,E},{B,E,F},{C,D,F}}.

The set of the straight lines D={{A},{B},{C},{D},{E},{F}}.

A point p∈ P and a straight line D∈ D, are incidental if and only if D∈ p.

The relations of parallelism and juxtaposition are those specified in definition 1 and 2.

The points of the model are non-neighboured two by twos. It can be noticed that {A}||{F},{B}||{D},{C}||{E}.

(A1) is true because through any two non-neighboured points of the model elapses only one straight line belonging to the model, for example through the points {A,B,C} and {A,D,E} elapses a unique straight line {A}, analogously it can be shown for the other points.

(A2) is true because from interpreting the chosen model, we conclude that each straight line contains two non-neighboured points.

(A3) The given model being finite, it can be easily verify that within it is verified the axiom (A3)

For example: let it be the point {B, C, A} and the straight line {E}, then there is a unique straight line {C}, in that way that {C}I{B,C,A} and {C}||{E}.

(A4) can be easily verified. For example: Let us take the straight lines {A} and {B}, they are crossed because {A}||{F} and {B}||{D} and {F} and {D} cuts each other in one single point {C,D,F}.

(A5) We take {B} and {E} crossed lines and let it be the point {E, A, D} positioned on the line {E} this point is non-neighboured to {B}∩{E}={B,E,F}, but {E, A, D} is non-neighboured to {B, C, A}, and the line which passes through {E, A, D} and cuts {B} it is the line {A} and we notice that it is crossed with {B}. So the axiom (A5) is verified.

3. THE MINIMALIST OF THE AXIOM STRUCTURE OF BARBILIAN AFFINE STRUCTURE

The minimalist of the system of the axioms consists in building for each axiom (Ai) i=1...5 a model Mi in which the axiom Ai is not satisfied, but the other axioms are not verified.

Model 1:

We will show_that within this model there are two non-neighboured points through which do not pass any line, so (A1) is not verified.

We will consider a square ABCD within the Euclidian plane (figure 2).

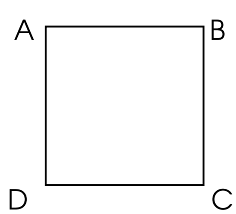

Figure 2.

The set of the points of the model P={{A},{B},{C},{D}}.

The set of the points D ={{A,B},{B,C},{C,D},{A,D}}.

A point p∈ P and a line D∈ D, are incident if only if p∈ D.

The relation of parallelism and neighbouring are those mentioned in the definitions 1 and 2. It can be noticed that the points of the model are two by two non-neighboured. The relation of parallelism is that from the Euclidian plan and it can be noticed {A,D}$\mid\mid${B,C}, and {A,B}$\mid\mid${D,C}.

(A1) is not verified because the points {A} and {C} are non-neighboured, but through them it does not pass any line.

(A2) is verified because each line of the model contains two not neighboured points.

(A3) The model being finite, axiom (A3) can be easily verified: for example for the point {A} and the line {D,C} there is a unique line {A,B}, which passes through {A} and it is parallel with the line {D,C}. (A4) is verified, for example: the line {A,B} is crossed with the line {A,D}, but also with the line {B,C}, it is the same for the other lines.

(A5) is true, if we take into consideration the lines {A,B} and {A,D} they are crossed, let it be {B}$\in${A,B}, it can be noticed that {B} is not neighboured to {A}={A,B}$\cap${A,D} but also through {B} there are not other lines which crosses the line {A,D} except {A,B} and {A,B}©{A,D}.

<u>Model 2</u>:

In the following model we will show that there is a point on a line which is neighboured to any other point on the line.

We will take a polyhedron in the Euclidian space with its apex A, B; C; D; E; F as in figure 3.

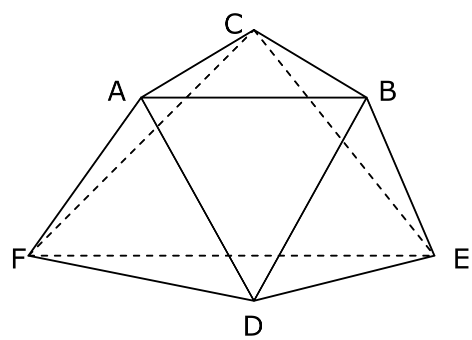

Figure 3.

The set of the points of the model is:
P={{A},{B},{C},{D},{D},{E},{F}}.

The set of the lines of the model there is:
D ={{A,B,D},{B,D,E},{C,E,F},{A,B,F},{B,C,E},{C,A,F},{A,E},{B,F},{C,D}}.

A point is incident with a line if the point belongs that line.

The relation of parallelism and non-neighbouring is mentioned in the definitions 1 and 2.

It can be noticed that the neighboured points are :{A} and {D}, {D} and {B}, {B} and {E}, {E} and {C}, {C} and {F}, {F} and {A}.

And the non-neighboured points are: {A} and {B}, {B} and {C}, {C} and {A}, {D} and {E}, {E} and {F}, {F} and {D}, {B} and {F}, {C} and {D}, {A} and {E}.

The parallel lines are: {A,B,D}$\mid\mid${C,E,F}, {B,E,D}$\mid\mid${A,F,C}, {C,B,E}$\mid\mid${A,F,D}, {A,E}$\mid\mid${C,D}$\mid\mid${B,F}.

(A1) is true because through the points {A} and {B} which are not neighboured passes a unique line, namely {A, B, D}.

$(\overline{A2})$ is satisfied because there is the point {D}$\in${A,B,D}, but the points {A} and {D} are neighboured, in the same way is {D} and {B}.

(A3) is true because for the point (A) and line {D,B,E} it can be assigned a unique line {A,C,F} through {A} parallel to {D,B,E}. Or, taking into consideration the point {A} and the line {C,D} through {A} it can be assigned a unique line {A,E} which is parallel to {C,D}.

(A4) is verified because for example: the line {A,B,D} is crossed with {B,C,E}, this results from the fact {A,B,D}$\|${C,E,F}, and {B,C,E}$\|${A,F,D}, but {C,F,E} cuts {A,F,D} in only one point {F}.

(A5) is verified, for example let it be {B,C,E} and {C,A,F} the two crossed lines and let it be the point {A}$\in${C,A,F}, we can notice that {A} is non-neighboured to {C}, where{C}={B,C,E}$\cap${C,A,F}. Through {A} passes:

1. the line {A,B,D} which cuts {B,C,E}, is crossed with {B,C,E} because the line {A,B,D}$\|${C,E,F} and {B,C,E}$\|${A,F,D} and {C,E,F}$\cap${A,F,D}={F}
2. the line {A,E}, which cuts {B,C,E}, is crossed with {B,C,E} because {A,E}$\|${C,D}$\|${F,B}, {B,C,E}$\|${A,F,D} and {C,D}$\cap${A,F,D}={D} or {F,B}$\cap${A,F,D}={F}.

<u>Model 3:</u>

In this model we will show that there is a pair (p,G)$\in$PxD, in that way that through p it is not possible to cross any line parallel to G

We will consider a polyhedron in the Euclidian space with apexes A,B,C,D,E as in figure 4.

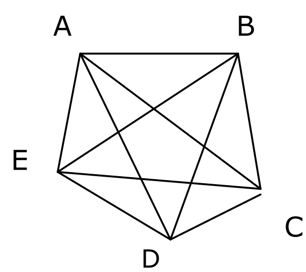

Figure 4.

The set of the points of the model is: P={{A},{B},{C},{D},{E}}.

The set of the lines of the model is:

D ={{A,B,E},{A,D,E},{A,C},{B,D},{B,C},{C,D},{E,C}}.

Neighboured points: AoE.

(A1) can be easily verified the model being finite.

(A2) is verified because the point B$\in${A,B,E} and BϕA, BϕE, D$\in${A,D,E} and DϕA, DϕE.

(A3) there is the line {E,C} and the point A, through A it can not be traced any parallel line to {E,C}, or choosing {B,D} and point C it can be assigned two parallel lines to {B,D} namely {A,C} and {E,C}.

(A4) the followings are verified:

{A,B,E}©{E,A,D} because {A,B,E}$\|${D,C} and {E,A,D}$\|${B,C} and {B,C}$\cap${D,C}={C}

{A,C}©{B,C} because {A,C}$\|${D,B} and {E,A,D}$\|${B,C} and {B,D}$\cap${E,A,D}={D}

{B,D}©{D,C} because {B,D}$\|${E,C} and {D,C}$\|${E,A,B} and {E,C}$\cap${E,A,B}={E} or {B,D}$\|${A,C} and {D,C}$\|${E,A,B} and {A,C}$\cap${E,A,B}={A}

{E,C}©{E,A,D} because {E,C}$\|${D,B} and {E,A,D}$\|${B,C} and {B,C}$\cap${D,B}={B}

(A5) {A,B,E}©{E,A,D}, D$\in${E,A,D}, {D}ϕ{A}, {D}ϕ{E} it can be noticed that {B,D} which cuts {E,A,D} then {B,D}©{E,A,D} because {B,D}$\|${E,C} and {E,A,B}$\|${B,C} and {E,D}$\cap${D,C}={C}, or {B,D}$\|${A,C} and {E,A,B}$\|${D,C} and {A,C}$\cap${D,C}={C}.

<u>Model 4.</u>

We will prove that there is a model within a line of the model is not crossed with any line of the model. We will take a cube in the Euclidian space with the apexes A,B,C,D,E,F,G,H (figure 5)

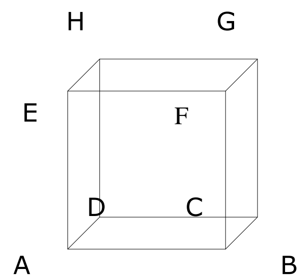

Figure 5.

The set of the points of the model is: P={{A},{B},{C},{D},{E},{F},{G},{H}}.

The set of the lines of the model is:
D ={{A,B,F,E},{B,C,G,F},{D,C,G,H},{A,D,H,E},{D,B,F,H},{A,C,G,E}}.

A point is incident with line if the point belongs to that line.

The relation of parallelism and non-neighbouring are mentioned in the definitions 1 and 2.

It can be noticed that the neighboured points are: {A} and {E}, {B} and {F}, {C} and {G}, {D} and {H}. And the parallel lines are: {A,D,H,E}$\|${B,C,G,F}, {A,E,F,B}$\|${D,C,H,G}, {A,C,G,E}$\|${D,B,F,H}.

From the interpretation of the model results:

(A1) is true because, for example {A} and {D} are non-neighboured and through them passes a unique line {A, D, H, E}, or if we will consider the points {E} and {C} through them will pass a unique line, namely {A, C, G, E}.

(A2) is verified because for example for the line {A,B,F,E}, the points {A} and {F} are non-neighboured, in the same way for {B} and {E}, {A} and {B}, {E} and {F} from the same line.

For the line {A,C,G,E}, the points {A} and {C} are non-neighboured, in the same for the points {A} and {G}. The points {D} and {B}, {H} and {F}, {B} and {H}, {D} and {F} are non-neighboured and through them passes a unique line, namely the line {H,D,B,F}.

(A3) is verified because for the pair ({A}, {D,B,F.H}) there is a unique line namely the line {A,C,G,E} in that way so {A}$\in${A,C,G,E} and {A,C,G,E}$\|${D,B,F,H}, and for ({A},{D,C,G,H}) there is a unique line {A,B,F,E} that way so {A}$\in${A,B,F,E} and {A,B,F,E}$\|${D,C,G,H}.

(A4) is not verified because the line {A,B,E,F} is not crossed by any other line. For example {A,B,E,F} is not crossed with {A,D,H,E} because {A,B,F,E}$\|${D,C,G,H} and {A,D,H,E}$\|${B,C,G,F}, and {D,C,G,H} crosses the line {B,C,G,F} in {C} and {G}.

It can be noticed that {A,B,E,F} is not crossed with {A,C,G,E} because {D,C, H,G} crosses the line {D,B,F,H} in two points {D} and {H}

It can be noticed that in this model all the lines are not crossed.

(A5) there are not crossed lines.

Model 5.

We will put in evidence a model in which there is a pair of crossed lines G and H, $H \cap G = q$ and for any $p \in H$, non-neighboured to q, there is a line F which passes through p and cuts G, but G is not crossed with F.

Let it be a hexagon A,B,C,D,E,F in the Euclidian plan.

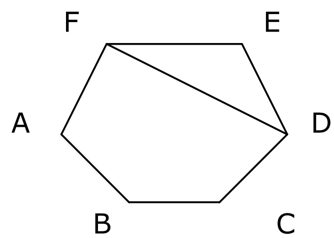

Figure 6.

The set of the points of the model is:

P= {{A},{B},{C},{D},{E},{F}}

The set of the lines of the model is:

D ={{F,A,B,C},{A,B,C,D},{E,A,B,C},{F,E},{E,D},{F,D}}

A point is incident to a line if the point belongs to that line.

The relation of parallelism and non-neighbouring are mentioned in the definitions 1 and 2.

It can be noticed that the neighboured points are: {A}o{B},{A}o{C},{B}o{C}.

And the parallel lines are: {E,A,B,C} $\|$ {F,D}, {F,A,B,C} $\|$ {E,D}, {A,B,C,D} $\|$ {F,E}.

The model being finite it can be noticed that the axiom (A1) is true. For example {A} ϕ {F} and through them passes a unique line {F,A,B,C}.

The axiom (A2) is verified, it results from the way the lines of the model were chosen.

For example choosing the line {F,A,B,C} and a point on it {A} it can be noticed that there is {F} $\in$ {F,A,B,C} and {F} ϕ {A}.

As a result of the interpretation of this finite model, taking into consideration that each line of the model has a unique parallel line, results the axiom (A3). For example the parallel line through {A} to {F,E} is the line {D,A,B,C}, the parallel through {A} to {D,E} is the line {F,A,B,C}, the parallel through {A} to {F,D} is the line {E,A,B,C}.

Axiom (A4) is verified, for example: {A,B,C,D}©{F,D} because {A,B,C,D} $\|$ {F,E} and {F,D} $\|$ {E,A,B,C} and {F,E} $\cap$ {E,A,B,C}={E}.

{A,B,C,E}©{E,D} because {A,B,C,E} $\|$ {F,D} and {E,D} $\|$ {F,A,B,C} and {F,D} $\cap$ {F,A,B,C}={F}.

{A,B,C,F}©{F,D} because {A,B,C,F} $\|$ {E,D} and {F,D} $\|$ {E,A,B,C} and {E,D} $\cap$ {E,A,B,C}={E}.

Axiom (A5) is not verified because choosing the lines {A,B,C,F} and {F,E} which are crossed and taking the point {E} $\in$ {F,E} it can be noticed that {E} ϕ {F}, where {F}={F,E} $\cap$ {A,B,C,F}.

Through the point {E} it can be taken the line {E,A,B,C} which intersects the line {A,B,C,F}, but they are not crossed because $\{E,A,B,C\} \cap \{A,B,C,F\} = \{A,B,C\}$.

4. CONCLUSIONS

The system of the axioms of the affine Barbilian structure is minimal, because it was demonstrated that each of its primary notions and relations is independent from the others and each axiom of the theory is independent from the others.

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