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ONE-DIMENSIONAL TRUNCATED von MISES DISTRIBUTION IN DATA MODELING

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ABSTRACT: Data modeling using the circular von Mises distribution can be tediously since the density function involves the modified Bessel function. In this paper, we have done some considerations on using the truncated distribution density instead of classic one, as the von Mises distribution can be adapted to possible situations in real cases. We obtained the truncated function preserving the properties of a probability density, accompanied by numerical examples.
KEYWORDS:
continuous distribution density, von Mises distribution, data modeling

1. THE ONE-DIMENSIONAL TRUNCATED VON MISES DISTRIBUTION LAW

The von Mises distribution is a well known symmetric and continuous distribution [1,2], whose density function is given by

$$f(x) = \frac{e^{k\cos(x-\mu)}}{2\pi I_0(k)} \tag{1}$$

where x belongs to a 2π length interval, I_0 denotes the modified Bessel function of the first kind and order 0, k and μ are two real parameters, with k>0 [3,4].

This paper introduces a computational method for a truncated modeling of the classic one-dimensional von Mises distribution.

The new function has to be null without a finite interval, which includes the mean value of the variable, and preserves the properties of probability density, so as the correlation coefficient of the new function should be as close as possible to one [6].

We propose to model the given points using the density function

$$ft(x) = \begin{cases} \frac{1}{K} \cdot \frac{e^{k\cos(x-\mu)}}{2\pi I_0(k)} , & t \in (\mu - \pi + \beta , \mu + \pi - \alpha) \\ 0 , & t \notin (\mu - \pi + \beta , \mu + \pi - \alpha) \end{cases} \tag{2}$$

where

$$K = \int_{\mu-\pi+\beta}^{\mu+\pi-\alpha} \frac{e^{k\cos(x-\mu)}}{2\pi I_0(k)} dx \tag{3}$$

and $\alpha > 0$, $\beta > 0$ are positive constants that are going to be determined.

The set of numeric data that are going to be processed is given bellow, where x represents the independent variable and y is the dependent one.

$x^T =$

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1.371	0.097	0.641	1.053	1.122	1.759	2.087	3.113	3.617	3.724	3.97	6.384

$y^T =$

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.013	0.11	0.178	0.252	0.266	0.366	0.375	0.224	0.147	0.134	0.111	0.022

These variables are characterized by

$$nv := \mathrm{length}(x)$$
$$i := 1..nv$$
$$mx := \mathrm{mean}(x) \quad mx = 2.18307$$
$$my := \mathrm{mean}(y) \quad my = 0.18312$$

where mx and my represent the mean values of the respective variables, and

$$sx := \mathrm{stdev}(x) \quad sx = 2.00188$$
$$sy := \mathrm{stdev}(y) \quad sy = 0.11274$$

where sx and sy are the standard deviations of variables x and y.

Let

$$variance := \frac{1}{nv-1}\cdot\sum_i \left(x_i - mx\right)^2 \tag{4}$$

be the variance of the variable x, and let

$$\mu := mx$$
$$k := \frac{1}{variance}$$
$$\mu = 2.18307$$
$$k = 0.22874$$

be the constants that appear in the von Mises density law.

With these values the classic law becomes

$$fc(x) := \frac{\exp\left(k\cdot\cos\left(x-\mu\right)\right)}{2\cdot\pi\cdot I0(k)} \tag{5}$$

having in our practical application the next table of values

$fc(x)^T =$

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.127	0.14	0.158	0.173	0.176	0.194	0.197	0.18	0.162	0.158	0.15	0.14

The function (5) has the graphic representation depicted in Figure 1, considering its 2π period.

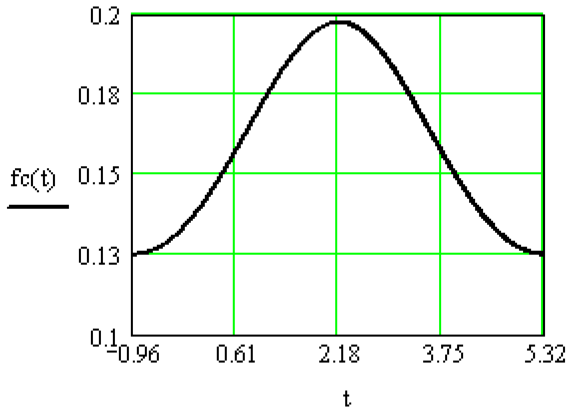

Figure 1. Probability density function for the classic von Mises law.

And, obviously, it is a non-null function that meets the condition

$$\int_{\mu-\pi}^{\mu+\pi} \frac{\exp(k \cdot \cos(x-\mu))}{2 \cdot \pi \cdot I0(k)} = 1 \tag{6}$$

2. THE METHOD OF TRUNCATED MODELING

Now we are going to introduce the truncated one-dimensional Mises density function, given by

$$ft(t) := \begin{cases} \dfrac{\dfrac{\exp(k \cdot \cos(t-\mu))}{2 \cdot \pi \cdot I0(k)}}{K} & \text{if } (\mu - \pi + \beta) < t < (\mu + \pi - \alpha) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where α and β are two positive constants that we are going to determine using the least squares method [5, 7].

In order to achieve this, the necessary condition is that the sum of the squares of the differences between the theoretical values of the function and the experimental values, be minimum.

That is, function

$$\Gamma(\alpha,\beta) := \sum_{i=1}^{nv} \left[\text{if}\left[(\mu - \pi + \beta) < x_i < (\mu + \pi - \alpha), \frac{\dfrac{\exp(k \cdot \cos(x_i-\mu))}{2 \cdot \pi \cdot I0(k)}}{K}, 0 \right] - y_i \right]^2 \tag{8}$$

equivalent with

$$\Gamma(\alpha,\beta) := \sum_{i=1}^{nv} \left[\begin{cases} \dfrac{\dfrac{\exp(k \cdot \cos(x_i-\mu))}{2 \cdot \pi \cdot I0(k)}}{K} & \text{if } (\mu - \pi + \beta) < x_i < (\mu + \pi - \alpha) \\ 0 & \text{otherwise} \end{cases} - y_i \right]^2 \tag{9}$$

where

$$K := \int_{\mu-\pi+\beta}^{\mu+\pi-\alpha} \frac{\exp(k\cdot\cos(x-\mu))}{2\cdot\pi\cdot I0(k)}\,dx \qquad (10)$$

shall be minimized with respect to α and β.

We shall find the minima of $\Gamma(\alpha,\beta)$ considering that $\alpha \in [\alpha i, \alpha s]$ and $\beta \in [\beta i, \beta s]$ with the help of the program we are giving hereinafter:

$$nt := 100 \quad \alpha i := 0 \quad \alpha s := 2 \quad \beta i := 0 \quad \beta s := 2$$

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prg := | fmin ← 10^10
       | for i ∈ 1 .. nt + 1
       |   | α_i ← αi + (i − 1)·(αs − αi)/nt
       |   | for j ∈ 1 .. nt + 1
       |   |   | β_j ← βi + (j − 1)·(βs − βi)/nt
       |   |   | inte_{i,j} ← ∫_{μ−π+β_j}^{μ+π−α_i} exp(k·cos(x − μ)) / (2·π·I0(k)) dx
       |   |   | f ← Σ_{ks=1}^{nv} [ ( (exp(k·cos(x_ks − μ)) / (2·π·I0(k))) / inte_{i,j}   if (μ − π + β_j) < x_ks < (μ + π − α_i)
       |   |   |                       0   otherwise ) − y_ks ]^2
       |   |   | if (f ≤ fmin)
       |   |   |   | αf ← α_i
       |   |   |   | βf ← β_j
       |   |   |   | fmin ← f
       | (αf, βf, fmin)
```

Using this program, we obtain the values of α and β that minimize the function (9) hereinbefore:

$\alpha f := prg_1$, $\alpha f = 1.08$

$\beta f := prg_2$, $\beta f = 0.8$

$fmin := prg_3$, $fmin = 0.05208$

These values of α and β substituted in expression (7) lead to the truncated one-dimensional Mises density function. In accordance to this modeling, the resulting value for K is

$$K := \int_{\mu-\pi+\beta f}^{\mu+\pi-\alpha f} \frac{\exp(k\cdot\cos(x-\mu))}{2\cdot\pi\cdot I0(k)}\,dx \qquad (11)$$

that is

$$K = 0.75676$$

The correlation coefficient of the truncated rule is

$$rtrunc := \sqrt{1-\frac{\sum_i \left[y_i - \begin{cases} \dfrac{\frac{\exp(k\cdot\cos(x_i-\mu))}{2\cdot\pi\cdot I0(k)}}{K} & \text{if } (\mu-\pi+\beta f) < x_i < (\mu+\pi-\alpha f) \\ 0 & \text{otherwise} \end{cases} \right]^2}{\sum_i (y_i - mean(y))^2}}$$

$$rtrunc = 0.8115 , \qquad (12)$$

and the deviation of the truncated rule is

$$\sigma trunc := \sqrt{\frac{\sum_i \left[y_i - \begin{cases} \dfrac{\frac{\exp(k\cdot\cos(x_i-\mu))}{2\cdot\pi\cdot I0(k)}}{K} & \text{if } (\mu-\pi+\beta f) < x_i < (\mu+\pi-\alpha f) \\ 0 & \text{otherwise} \end{cases} \right]^2}{nv}}$$

$$\sigma trunc = 0.06588 \qquad (13)$$

In the case of the classic von Mises probability density

$$fc(x) := \frac{\exp(k\cdot\cos(x-mx))}{2\cdot\pi\cdot I0(k)} , \quad fc(x) := \frac{\exp(0.22874\cdot\cos(x-2.18307))}{2\cdot\pi\cdot I0(0.22874)} \qquad (14)$$

the correlation coefficient is

$$rcl := \sqrt{1-\frac{\sum_i \left(y_i - \frac{\exp(k\cdot\cos(x_i-mx))}{2\cdot\pi\cdot I0(k)}\right)^2}{\sum_i (y_i - mean(y))^2}}$$

$$rcl = 0.53923 \qquad (15)$$

and further, it has the deviation

$$\sigma cl := \sqrt{\frac{\left[\sum_i \left(y_i - \frac{\exp(k\cdot\cos(x_i - mx))}{2\cdot\pi\cdot I0(k)}\right)^2\right]}{nv}}$$

$$\sigma cl = 0.09495 \qquad (16)$$

The graphic representation of the truncated probability density

$$ft(t) := \begin{cases} \dfrac{\dfrac{\exp(0.22874\cdot\cos(t-2.18307))}{2\cdot\pi\cdot I0(0.22874)}}{0.75676} & \text{if } (2.18307-\pi+0.8) < t < (2.18307+\pi-1.08) \\ 0 & \text{otherwise} \end{cases} \qquad (17)$$

is depicted in Figure 2, comparatively with the classic case.

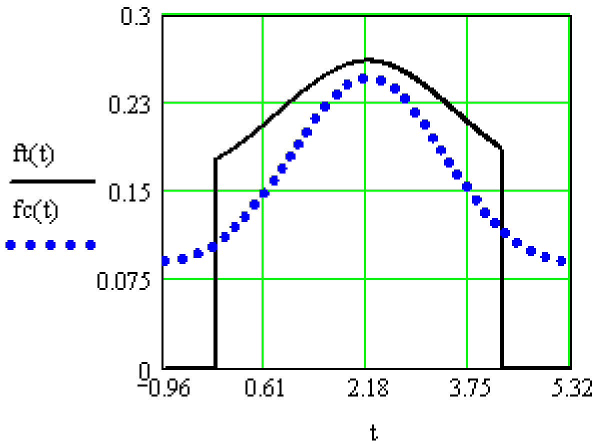

Figure 2. Probability density functions for the truncated von Mises law and for the classic von Mises law, depicted comparatively.

The cumulative distribution function of the truncated density

$$Ftr(t) := \begin{cases} \displaystyle\int_{\mu-\pi+\beta f}^{t} \frac{\dfrac{\exp(k\cdot\cos(xv-\mu))}{2\cdot\pi\cdot I0(k)}}{K}\,dxv & \text{if } (\mu-\pi+\beta f) < t < (\mu+\pi-\alpha f) \\ 1 & \text{if } t \ge \mu+\pi-\alpha f \\ 0 & \text{if } t \le \mu-\pi+\beta f \end{cases} \qquad (18)$$

comparatively with the classic case

$$Fcl(t) := \begin{cases} \displaystyle\int_{\mu-\pi}^{t} \frac{\exp(k\cdot\cos(xv-mx))}{2\cdot\pi\cdot I0(k)}\,dxv \\ 1 & \text{if } t \ge \mu+\pi \\ 0 & \text{if } t \le \mu-\pi \end{cases} \qquad (19)$$

is represented in Figure 3.

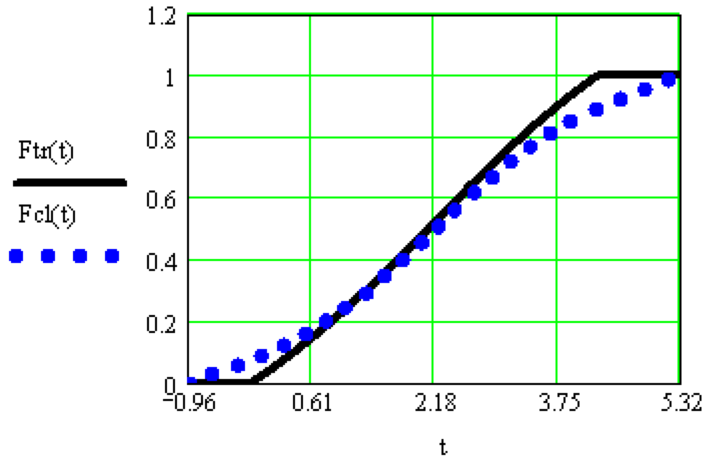

Figure 3. The cumulative distribution function of the truncated von Mises law against the cumulative distribution function of the classic law.

The characteristic function of the truncated modeling is

$$ex(t) := \int_{\mu-\pi}^{\mu+\pi} e^{1i \cdot t \cdot xv} \cdot \begin{cases} \dfrac{\dfrac{\exp(k \cdot \cos(xv-\mu))}{2 \cdot \pi \cdot I0(k)}}{K} & \text{if } (\mu - \pi + \beta f) < xv < (\mu + \pi - \alpha f) \\ 0 & \text{otherwise} \end{cases} \, dxv \quad (20)$$

having its components represented in Figure 4.

$$t := \mu - \pi, \mu - \pi + 0.05 .. \mu + \pi$$

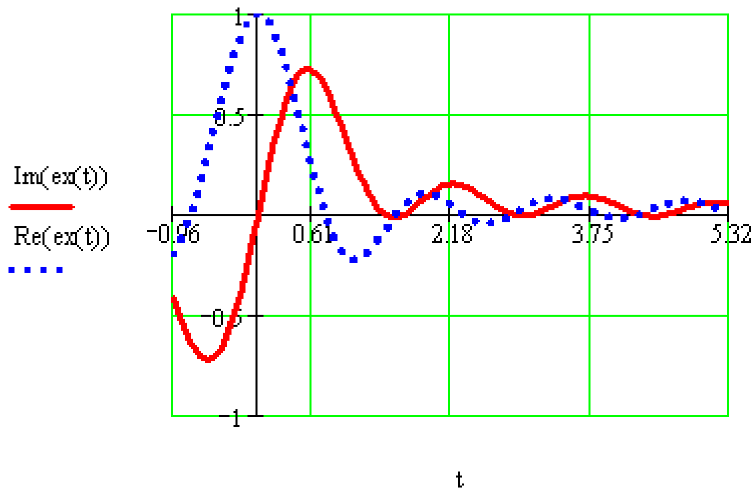

Figure 4. The components of the characteristic function of one-dimensional truncated Mises law.

3. CONCLUSIONS

The considerations presented in the paper may lead to the conclusion that the rule of truncated von Mises distribution obtained under the given conditions, preserving the properties of a probability density, leads to a higher correlation coefficient and to a smaller deviation than in the case of the classic distribution, as it can be noticed from the results given hereinafter:

$$rcl = 0.53923 \quad < \quad rtrunc = 0.8115$$

$$\sigma cl = 0.09495 \quad > \quad \sigma trunc = 0.06588$$

Therefore, the optimal modeling of the existent data can be obtained by means of the truncated distribution. In practical cases the use of truncated modeling can be more expedient instead of using the classic law.

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