JOURNAL OF ENGINEERING
annals OF
FACULTY OF ENGINEERING HUNEDOARA
TOME VI (year 2008), FASCICULE 3, (ISSN 1584 – 2673)

FINITE ELEMENT ALGORITHM FOR VELOCITY PROFILE OF A LAMINAR BOUNDED FLOW

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ABSTRACT: The authors have conceived a computational algorithm for solving the velocity field during the motion of an incompressible fluid between two parallel plates, using the Finite Element Method. The paper sets the theoretical description of the finite elemet method applied in fluid dynamics, giving in the second part a friendly algorithm that calculates the velocity profile, written in MathCAD. The stream is considered having very small Reynolds numbers, as laminar flow.

KEYWORDS:
Computational fluid dynamics, Laminar flow, Finite element method.

1. FLOW EQUATIONS, BOUNDARY CONDITIONS AND FEM DISCRETIZATION

We consider the motion of an incompressible fluid between two parallel plates. The upper plate moves with constant horizontal speed, generating a one-way stream, having very small Reynolds numbers (laminar flow).

The governing equation of motion is

$$\frac{dp}{dx} = \mu \frac{d^2u}{dy^2}, \tag{1}$$

where p is the pressure, u – the fluid velocity and μ – dynamic viscosity coefficient.

This equation can be written in the following more convenient form

$$\frac{d^2u^*}{dy^{*2}} + P^* = 0, \tag{2}$$

using followed dimensionless variables

$$u^* = \frac{u}{U_0}, \quad y^* = \frac{y}{h}, \quad P^* = \frac{h^2}{\mu U_0}\left(-\frac{dp}{dx}\right), \tag{3}$$

where U_0 is the upper plate velocity and h being the height between plates (see Fig.1).

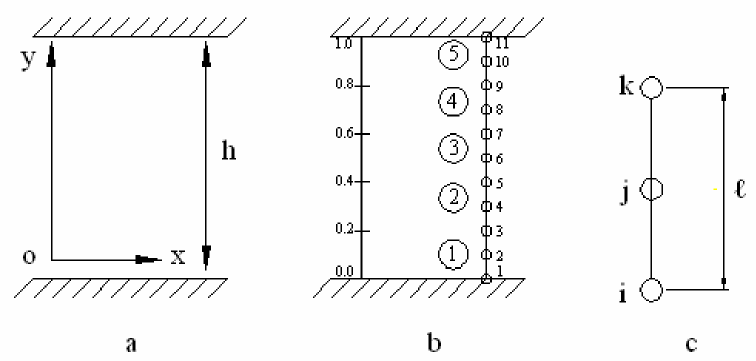

Fig.1 **a** – System coordinates; **b** – FEM discretization; **c** – Three nodes finite element.

The boundary conditions needed for computing the flow within a solution domain are

$$u^* = 0 \quad \text{for} \quad y^* = 0$$
$$u^* = 1 \quad \text{for} \quad y^* = 1. \tag{4}$$

2. REQUIREMENTS ON FINITE ELEMENT SHAPE FUNCTIONS

In order to obtain the expressions of the element shape functions, we consider the continuous function

$$u(y) = \alpha_1 + \alpha_2 y + \alpha_3 y^2. \tag{5}$$

When written out in component form, results the system as below

$$\begin{cases} \alpha_1 + \alpha_2 y_i + \alpha_3 y_i^2 = u_i \\ \alpha_1 + \alpha_2 y_j + \alpha_3 y_j^2 = u_j \\ \alpha_1 + \alpha_2 y_k + \alpha_3 y_k^2 = u_k \end{cases}. \tag{6}$$

For a finite element e, which can be seen in Figure 1c, having the length ℓ and three nodes i, j, k, let us consider

$$y_i = 0, \quad y_j = \frac{\ell}{2}, \quad y_k = \ell \tag{7}$$

and substitution of these in system (6) lead to the next results

$$\alpha_1 = u_i, \quad \alpha_2 = \frac{1}{\ell}\left(-3u_i + 4u_j - u_k\right), \quad \alpha_3 = \frac{2}{\ell^2}\left(u_i - 2u_j + u_k\right). \tag{8}$$

Including relations (8) in equation (5) lead to expression

$$u = \left(1 - \frac{2y^*}{\ell}\right)\left(1 - \frac{y^*}{\ell}\right)u_i + \frac{4y^*}{\ell}\left(1 - \frac{y^*}{\ell}\right)u_j + \frac{y^*}{\ell}\left(\frac{2y^*}{\ell} - 1\right)u_k \tag{9}$$

and the element shape functions are

$$N_i(y^*) = \left(1 - \frac{2y^*}{\ell}\right)\left(1 - \frac{y^*}{\ell}\right), \quad N_j(y^*) = \frac{4y^*}{\ell}\left(1 - \frac{y^*}{\ell}\right), \quad N_k(y^*) = -\frac{y^*}{\ell}\left(1 - \frac{2y^*}{\ell}\right). \tag{10}$$

We can now approximate the velocity on a finite element domain by function

$$\hat{u} = N_i(y^*)u_i + N_j(y^*)u_j + N_k(y^*)u_k, \tag{11}$$

which satisfies equation (2). Following calculations in the Galerkin's method, we obtained the governing equations in matrix form, for the components of fluid velocity, considering the stream crossing a single finite element

$$\begin{bmatrix} \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_i}{dy^*}dy^* & \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_j}{dy^*}dy^* & \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_k}{dy^*}dy^* \\ \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_i}{dy^*}dy^* & \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_j}{dy^*}dy^* & \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_k}{dy^*}dy^* \\ \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_i}{dy^*}dy^* & \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_j}{dy^*}dy^* & \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_k}{dy^*}dy^* \end{bmatrix} \cdot \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = \begin{bmatrix} \int_0^\ell N_i P^* dy^* - \left(\frac{d\hat{u}}{dy^*}\right)_{y_i^*} \\ \int_0^\ell N_j P^* dy^* \\ \int_0^\ell N_k P^* dy^* + \left(\frac{d\hat{u}}{dy^*}\right)_{y_k^*} \end{bmatrix} \tag{12}$$

Differentiate the velocity function (11) we get the followed relations

$$\left(\frac{d\hat{u}}{dy^*}\right)_{y_i^*} = \left.\frac{dN_i}{dy^*}\right|_{y_i^*}\cdot u_i + \left.\frac{dN_j}{dy^*}\right|_{y_i^*}\cdot u_j + \left.\frac{dN_k}{dy^*}\right|_{y_i^*}\cdot u_k$$
$$\left(\frac{d\hat{u}}{dy^*}\right)_{y_k^*} = \left.\frac{dN_i}{dy^*}\right|_{y_k^*}\cdot u_i + \left.\frac{dN_j}{dy^*}\right|_{y_k^*}\cdot u_j + \left.\frac{dN_k}{dy^*}\right|_{y_k^*}\cdot u_k \quad (13)$$

which can be substitute in (12), giving the matrix equation for a finite element

$$k^{<e>}\cdot u^{<e>} = F^{<e>}\ , \quad (14)$$

where

$$k^{<e>} = \begin{bmatrix} \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_i}{dy^*}dy^* + \left.\frac{dN_i}{dy^*}\right|_{y_i^*} & \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_j}{dy^*}dy^* + \left.\frac{dN_j}{dy^*}\right|_{y_i^*} & \int_0^\ell \frac{dN_i}{dy^*}\frac{dN_k}{dy^*}dy^* + \left.\frac{dN_k}{dy^*}\right|_{y_i^*} \\ \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_i}{dy^*}dy^* & \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_j}{dy^*}dy^* & \int_0^\ell \frac{dN_j}{dy^*}\frac{dN_k}{dy^*}dy^* \\ \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_i}{dy^*}dy^* - \left.\frac{dN_i}{dy^*}\right|_{y_k^*} & \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_j}{dy^*}dy^* - \left.\frac{dN_j}{dy^*}\right|_{y_k^*} & \int_0^\ell \frac{dN_k}{dy^*}\frac{dN_k}{dy^*}dy^* - \left.\frac{dN_k}{dy^*}\right|_{y_k^*} \end{bmatrix},$$

$$\left(u^{<e>}\right)^T = \begin{bmatrix} u_i & u_j & u_k \end{bmatrix}; \qquad \left(F^{<e>}\right)^T = \begin{bmatrix} \int_0^\ell N_i P^* dy^* & \int_0^\ell N_j P^* dy^* & \int_0^\ell N_k P^* dy^* \end{bmatrix}.$$

3. FINITE ELEMENT ALGORITHM FOR VELOCITY FIELD

Having the matrix equation that defines the velocity components along a finite element, we are now able to assembly the system matrix equation, that describes the velocity profile along the FEM discretization (see Fig.1b). The next algorithm, written using MathCad computer algebra system, afford the facility to introduce a desired number of finite elements and also, a conceivable value of pressure gradient along x.

$ORIGIN \equiv 1$

Value of P:

$P := 10$

Number of finite elements:

$n := 4$

Nodes:

$nnod := 2\cdot n + 1$

Length of an element:

$$lu := \frac{1}{n}$$

Element shape functions:

$$N1(y) := \left(1 - 2\cdot\frac{y}{lu}\right)\cdot\left(1 - \frac{y}{lu}\right) \quad N2(y) := 4\cdot\frac{y}{lu}\cdot\left(1 - \frac{y}{lu}\right) \quad N3(y) := \frac{-y}{lu}\cdot\left(1 - 2\cdot\frac{y}{lu}\right)$$

$$k_{1,1} := \left[\int_0^{lu}\left(\frac{d}{dy}N1(y)\right)\cdot\left(\frac{d}{dy}N1(y)\right)dy\right] \qquad k_{1,2} := \int_0^{lu}\left(\frac{d}{dy}N1(y)\right)\cdot\left(\frac{d}{dy}N2(y)\right)dy$$

$$k_{1,3} := \int_0^{lu} \left(\frac{d}{dy}N1(y)\right)\cdot\left(\frac{d}{dy}N3(y)\right) dy \qquad k_{2,2} := \int_0^{lu} \left(\frac{d}{dy}N2(y)\right)\cdot\left(\frac{d}{dy}N2(y)\right) dy$$

$$k_{2,3} := \int_0^{lu} \left(\frac{d}{dy}N2(y)\right)\cdot\left(\frac{d}{dy}N3(y)\right) dy \qquad k_{3,3} := \int_0^{lu} \left(\frac{d}{dy}N3(y)\right)\cdot\left(\frac{d}{dy}N3(y)\right) dy$$

$$k_{2,1} := k_{1,2} \quad k_{3,1} := k_{1,3} \quad k_{3,2} := k_{2,3}$$

Elemental matrix:

$$k = \begin{pmatrix} 9.33333333333327 & -10.666666666666623 & 1.333333333333321 \\ -10.666666666666623 & 21.333333333333336 & -10.666666666666671 \\ 1.333333333333321 & -10.666666666666671 & 9.3333333333333334 \end{pmatrix}$$

Constructing the global system matrix:

```
KF :=  | for  i ∈ 1 ..  nnod
       |   for  j ∈ 1 ..  nnod
       |     KF[i,j] ← 0
       | KF

KFE := | KFE ← KF
       | p ← 0
       | for  e ∈ 1 ..  n
       |   | for  i ∈ (2·e − 1) ..  (2·e + 1)
       |   |   for  j ∈ (2·e − 1) ..  (2·e + 1)
       |   |     KFE[i,j] ← KFE[i,j] + k[i−p,j−p]
       |   | p ← p + 2
       | KFE
```

MC := KFE

$$MC = \begin{pmatrix} 9.333 & -10.667 & 1.333 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10.667 & 21.333 & -10.667 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.333 & -10.667 & 18.667 & -10.667 & 1.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.667 & 21.333 & -10.667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.333 & -10.667 & 18.667 & -10.667 & 1.333 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.667 & 21.333 & -10.667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.333 & -10.667 & 18.667 & -10.667 & 1.333 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10.667 & 21.333 & -10.667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.333 & -10.667 & 9.333 \end{pmatrix}$$

Elemental column vector of the right hand side of equation (14):

$$te := \begin{pmatrix} \int_0^{lu} N1(y)\cdot P\, dy \\ \int_0^{lu} N2(y)\cdot P\, dy \\ \int_0^{lu} N3(y)\cdot P\, dy \end{pmatrix} \qquad te = \begin{pmatrix} 0.416666666666667 \\ 1.666666666666667 \\ 0.416666666666667 \end{pmatrix}$$

Assembly the global vector of the right hand side of equation (14):

```
TL :=  | for  i ∈ 1 .. nnod
       |     TL_i ← 0
       | TL

TLE := | TLE ← TL
       | p ← 0
       | for  e ∈ 1 .. n
       |     | for  i ∈ (2·e − 1) .. (2·e + 1)
       |     |     TLE_i ← TLE_i + te_(i−p)
       |     | p ← p + 2
       | TLE
```

Boundary conditions:

$TLE_1 := 0$

$TLE_{nnod} := 1$

$TLE_{nnod-2} := -MC_{nnod-2,\,nnod} + TLE_{nnod-2}$

$TLE_{nnod-1} := -MC_{nnod-1,\,nnod} + TLE_{nnod-1}$

Review the system column vector

Without boundary conditions:

$$TLE = \begin{pmatrix} 0.416666666666667 \\ 1.666666666666667 \\ 0.833333333333333 \\ 1.666666666666667 \\ 0.833333333333333 \\ 1.666666666666667 \\ 0.833333333333333 \\ 1.666666666666667 \\ 0.416666666666667 \end{pmatrix}$$

Applying boundary conditions:

$$TLE = \begin{pmatrix} 0 \\ 1.666666666666667 \\ 0.833333333333333 \\ 1.666666666666667 \\ 0.833333333333333 \\ 1.666666666666667 \\ -0.499999999999987 \\ 12.333333333333337 \\ 1 \end{pmatrix}$$

$MFC := MC$ $\quad i := 2..nnod$ $\quad j := 1..(nnod-1)$

$MFC_{1,1} := 1$ $\quad MFC_{1,i} := 0$ $\quad MFC_{nnod,j} := 0$

$MFC_{nnod,nnod} := 1$ $\quad MFC_{i,1} := 0$ $\quad MFC_{j,nnod} := 0$

The system matrix with boundary conditions applied:

$$MFC = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 21.333 & -10.667 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10.667 & 18.667 & -10.667 & 1.333 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10.667 & 21.333 & -10.667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.333 & -10.667 & 18.667 & -10.667 & 1.333 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10.667 & 21.333 & -10.667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.333 & -10.667 & 18.667 & -10.667 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10.667 & 21.333 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$dim^T = (0 \;\; 0.125 \;\; 0.25 \;\; 0.375 \;\; 0.5 \;\; 0.625 \;\; 0.75 \;\; 0.875 \;\; 1)$$

```
dim := | dim_1 ← 0
       | for  k ∈ 2 .. nnod
       |     dim_k ← (k − 1) · 1/(nnod − 1)
       | dim
```

„dim" is a column vector that express the height coordinate of each node.

Finally, calculate the velocity field:

$$v := MFC^{-1} \cdot TLE \qquad v = \begin{pmatrix} 0 \\ 0.671875000000003 \\ 1.187500000000005 \\ 1.546875000000004 \\ 1.750000000000007 \\ 1.796875000000003 \\ 1.687500000000006 \\ 1.421874999999999 \\ 1 \end{pmatrix}$$

4. RESULTS AND GRAPHIC ANALYSIS

Computing the velocity field for several different numbers of finite elements, at different values of the pressure we obtained the velocity profiles depicted in the next figures.

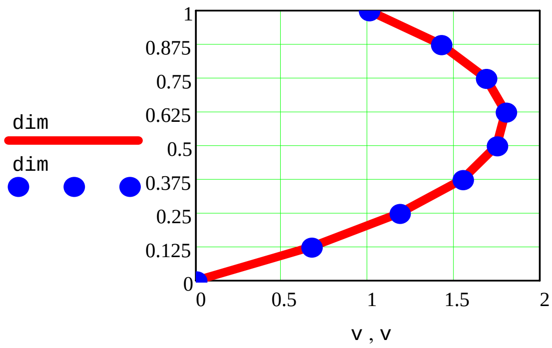

Fig.2 Velocity profile for 4 elements, for a negative pressure gradient (dp/dx<0), at $P = 10$.

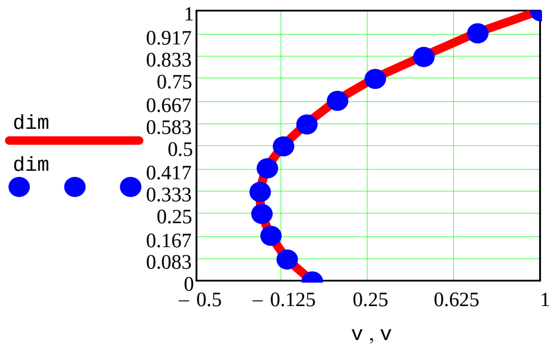

Fig.3 Velocity profile for 6 elements, for a positive pressure gradient (dp/dx>0), at $P = -5$.

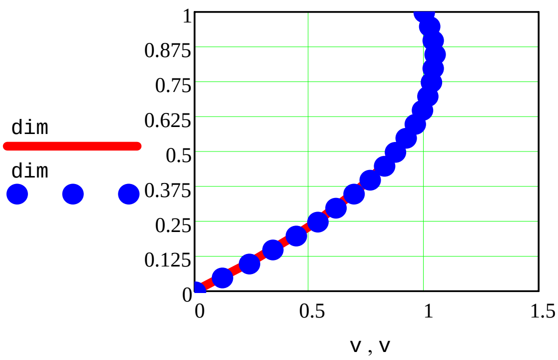

Fig.4 Velocity profile for 10 elements, for a negative pressure gradient (dp/dx<0), at $P = 3$.

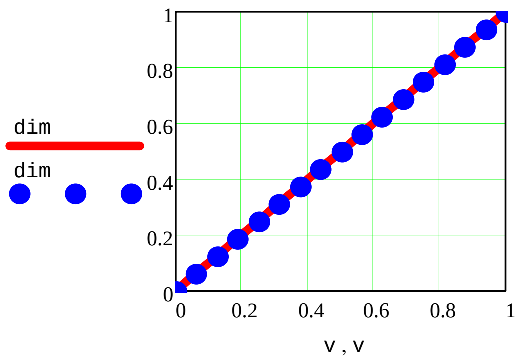

Fig.5 Linear velocity distribution (shear flow), obtained at no pressure gradient $P = 0$, for 8 elements.

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