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BOUNDS ON THE ACTUARIAL PRESENT VALUE OF GROUP LIFE INSURANCE

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ABSTRACT
In this paper we will show how one can apply known probability bounding techniques for bounding the actuarial present value of group life insurance. We will regard the probability bounding techniques by aggregation and disaggregation in linear programs published by Prékopa and Gao (see [2]).
There will be shown in the paper that these probability bounding techniques can be applied in the framework of the actuarial present value of group life insurance. Numerical test results will also be presented.
KEYWORDS
life insurance, probability bounding techniques, Numerical test results

1. INTRODUCTION

In an earlier paper Prékopa and Horváth (see [3]) applied the so called binomial moment lower and upper bounds for constructing bounds on the actuarial present value of group life insurance. The binomial moment probability bounds can be obtained by the solution of linear programs according to the binomial moments. These linear programs may be called aggregated programs as they utilize some lower order product event probabilities in aggregated form, namely the given order product probabilities are summed up (aggregated) in the binomial moment values. Recently Prékopa and Gao (see [2]) pointed out that the individual product event probabilities can be utilized in the linear programs without aggregation, too. The linear programs formulated this way may be called disaggregated linear programs and it is easy to prove that their optimal solutions provide better bounds on the probability of union of events. In this paper we will show that by the application these better probability bounds one can get better bounds on the actuarial present value of group life insurance, too. In the second section of the paper we briefly describe the concept of the actuarial present value of group life insurance and show how the probability bounds can be applied for bounding their value, in general. In the third section the probability bounding techniques by aggregation and disaggregation in linear programs will be described as it was worked out by Prékopa and Gao (see [2]). In the fourth section some numerical test results will be given, proving the superiority of the applied new probability bounding techniques.

2. THE ACTUARIAL PRESENT VALUE OF GROUP LIFE INSURANCE

In this section we will use the concepts and terminologies introduced by Bowers in his book (see [1]). The life time of a person is random variable, let it be designated by X. If the person lived x years then his remaining life time will be designated by $T(x)$, which is also a random variable. The probability distribution of $T(x)$ can be expressed by the conditional probability distribution of X. Indeed, if we denote by $F(x)$ the probability distribution function of X:

$$F(x)=P(X\le x),\ x\ge 0 \tag{2.1}$$

and introduce the notation:

$$s(x)=1-F(X), \tag{2.2}$$

then the probability distribution function of $T(x)$ can be given in the following way:

$$\begin{aligned}{}_t q_x &= P(T(x)\le t)=P(x<X\le x+t \mid X>x)=\frac{P(x<X\le x+t)}{P(X>x)}\\ &=\frac{F(x+t)-F(x)}{s(x)}=\frac{s(x)-s(x+t)}{s(x)}.\end{aligned} \tag{2.3}$$

Further notations will be:

$$\begin{aligned}{}_t p_x &= P(T(x)>t),\\ p_x &= {}_1 p_x,\\ q_x &= {}_1 q_x.\end{aligned}$$

Let us regard n persons in their ages $x_1,\dots,x_n$ years, where the numbers $x_1,\dots,x_n$ are not necessarily integers. As the time is going on, the number of survivors will decrease. Let be designated by

$$\left(\frac{k}{x_1x_2\cdots x_n}\right)$$

if the number of survivors out of n persons with ages $x_1,\dots,x_n$ is at least k and by

$$\left(\frac{[k]}{x_1x_2\cdots x_n}\right)$$

if the number of survivors out of n persons with ages $x_1,\dots,x_n$ is exactly k.

Then the probability that after t years from present at least k survivors exist will be designated as

$${}_t p_{\frac{k}{x_1x_2\cdots x_n}}$$

and similarly the probability that after t years from present exactly k survivors exist will be designated as

$${}_t p_{\frac{[k]}{x_1x_2\cdots x_n}}.$$

It is well known in the probability theory that for any $c_0,c_1,\dots,c_n$ and $d_0,d_1,\dots,d_n$ numbers

$$\sum_{k=0}^{n} c_k\, {}_t p_{\frac{[k]}{x_1,x_2,\dots,x_n}} = c_0+\sum_{k=1}^{n} {}_t S_k \Delta^k c_0 \tag{2.4}$$

and

$$\sum_{k=0}^{n} d_k\, {}_t p_{\frac{[k]}{x_1,x_2,\dots,x_n}} = d_0+\sum_{k=1}^{n} {}_t S_k \Delta^{k-1} d_1, \tag{2.5}$$

where

$${}_t S_k = \sum_{1\le i_1<\dots<i_k\le n} P\left(T(x_{i_1})>t,\dots,T(x_{i_k})>t\right) \tag{2.6}$$

and

$$\Delta^k c_0=\sum_{i=0}^{k}(-1)^{i-1}\binom{k}{i}c_i$$

is the kth difference of the series $c_0,c_1,\dots,c_n$.

Some types of actuarial present values can be calculated as the numerical integral over the time horizon of the left hand sides in the formulae (2.4) and (2.5). We can do the same with the right hand sides of the formulae (2.4) and (2.5), however the main numerical problem will be the calculation of the probability values involved in the expression of ${}_tS_k$ in formula (2.6). If the random variables $T(x_1),\dots,T(x_n)$ are independent, then

$$P\left(T\left(x_{i_1}\right)>t,\dots,T\left(x_{i_k}\right)>t\right)=P\left(T\left(x_{i_1}\right)>t\right)\cdots P\left(T\left(x_{i_k}\right)>t\right)$$

and we need the $P\left(T\left(x_{i_1}\right)>t,\dots,T\left(x_{i_k}\right)>t\right)$ probability values only for $k=1$. However they are dependent random variables, we may be able to calculate these probabilities for relatively small values of *k*, and then we will be able to utilize the probability bounding techniques given in the next section.

3. PROBABILITY BOUNDING TECHNIQUES BY AGGREGATION AND DISAGGREGATION IN LINEAR PROGRAMS

In this section we will follow the train of thought applied in the paper by Prékopa and Gao ([2]). Let us regard the classical inclusion-exclusion formula giving the probability of the union of events $A_1,\dots,A_n$ in terms of the intersection probabilities of the same events:

$$P(A_1\cup\cdots\cup A_n)=S_1-S_2+\cdots+(-1)^{n-1}S_n,$$

where $S_k=\sum_{1\le i_1<\cdots<i_k\le n}P\left(A_{i_1}\cap\cdots\cap A_{i_k}\right),\ k=1,\dots,n.$

If we know all $S_k,\ k=1,\dots,n$ values, then the probability of the union of events $A_1,\dots,A_n$ is determined by the above inclusion-exclusion formula. If, however, we only know $S_1,\dots,S_m$, where $m<<n$, then the most what we can do is to look for lower and upper bounds on the probability of the union. Let $S_0=1$ by definition and designate ν the number of those events (among $A_1,\dots,A_n$) which occur, then we have the relations

$$S_k=E\left[\binom{\nu}{k}\right]=\sum_{i=0}^{n}\binom{i}{k}v_i,\ k=0,1,\dots,n,$$

where $v_i=P(\nu=i),\ i=0,1,\dots,n.$ As we have trivially the equality $P(A_1\cup\cdots\cup A_n)=v_1+\cdots+v_n$, to obtain lower and upper bounds on the probability of the union we can formulate two linear programs:

$$\begin{aligned}&\min(\max) && \sum_{i=1}^{n}v_i\\ &\text{s.t.} && \sum_{i=1}^{n}\binom{i}{k}v_i=S_k,\quad k=1,\dots,m,\quad v_i\ge 0,\ i=1,\dots,n\end{aligned}\qquad(3.1)$$

Problem (3.1) use the probabilities $p_{i_1\dots i_k}=P\left(A_{i_1}\cap\cdots\cap A_{i_k}\right)$ in aggregated form, i.e., $S_1,\dots,S_m$ are used rather than the probabilities in these sums. This is why we call (3.1) aggregated problems.

The linear programs which make us possible to use the probabilities $p_{i_1\dots i_k}$, $1\le i_1<\cdots<i_k\le n$ individually, will be called disaggregated, and can be formulated as follows. Define

$$a_{IJ}=\begin{cases}1, & \text{if } I\subset J,\\ 0, & \text{if } I\not\subset J,\end{cases}$$

$$t_J=P\left(\left(\bigcap_{j\in J}A_j\right)\cap\left(\bigcap_{j\notin J}\overline{A}_j\right)\right)$$

$$p_I=P\left(\bigcap_{j\in I}A_j\right)$$

for any $I, J \subset \{1, \ldots, n\}$. Then we have the equation

$$\sum_{J \subset \{1,\ldots,n\}} a_{IJ} t_J = p_I, \quad I \subset \{1, \ldots, n\}.$$

Now we can formulate the following two linear programs, called disaggregated programs:

$$\begin{array}{lll} \min(\max) & \sum\limits_{0 \neq J \subset \{1,\ldots,n\}} t_J & \\ \text{s.t.} & \sum\limits_{0 \neq J \subset \{1,\ldots,n\}} a_{IJ} t_J = p_I, & |I| \le m, \quad t_J \ge 0, \quad J \subset \{1, \ldots, n\} \end{array} \tag{3.2}$$

We remark that Prékopa and Gao ([2]) defined a third type of LP by the use of partial aggregation/disaggregation of the LPs (3.1) and (3.2) and presented new bounds for the union of events based on the new type LP. In our case we will be able to solve the fully disaggregated LP problem (3.2), so we will not apply partially aggregated/disaggregated LPs.

4. NUMERICAL TEST RESULTS

In our test example we took a group of four persons (father, mother son and daughter with given ages) and took their surviving probabilities from real statistical data. The joint surviving probabilities were constructed such a way that the correlation be as large as possible and in another example opposite way as small as possible. The sequence of real numbers were $c = \frac{1}{2}; c_0 = 0, c_1 = \frac{1}{8}, c_2 = \frac{1}{4}, c_3 = \frac{1}{2}, c_4 = 1.$

In this case the aggregated LP was:

$$\begin{array}{llllll} \min(\max) \; (& \frac{1}{8} v_1 & + \frac{1}{4} v_2 & + \frac{1}{2} v_3 & + v_4) & \\ v_0 & + v_1 & + v_2 & + v_3 & + v_4 & = 1 (= S_0) \\ & v_1 & + 2v_2 & + 3v_3 & + 4v_4 & = S_1 \\ & & v_2 & + 3v_3 & + 6v_4 & = S_2 \\ v_0 \ge 0, & v_1 \ge 0, & v_2 \ge 0, & v_3 \ge 0, & v_4 \ge 0 & \end{array}$$

and the disaggregated LP was:

$$\min(\max) \; (\tfrac{1}{8}t_1 + \tfrac{1}{8}t_2 + \tfrac{1}{8}t_3 + \tfrac{1}{8}t_4 + \tfrac{1}{4}t_5 + \tfrac{1}{4}t_6 + \tfrac{1}{4}t_7 + \tfrac{1}{4}t_8 + \tfrac{1}{4}t_9 + \tfrac{1}{4}t_{10} + \tfrac{1}{2}t_{11} + \tfrac{1}{2}t_{12} + \tfrac{1}{2}t_{13} + \tfrac{1}{2}t_{14} + t_{15})$$

$$\begin{aligned}
t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9 + t_{10} + t_{11} + t_{12} + t_{13} + t_{14} + t_{15} &= 1 \\
t_1 + t_5 + t_6 + t_7 + t_{11} + t_{12} + t_{13} + t_{15} &= p_1 \\
t_2 + t_5 + t_8 + t_9 + t_{11} + t_{12} + t_{14} + t_{15} &= p_2 \\
t_3 + t_6 + t_8 + t_{10} + t_{11} + t_{13} + t_{14} + t_{15} &= p_3 \\
t_4 + t_7 + t_9 + t_{10} + t_{12} + t_{13} + t_{14} + t_{15} &= p_4 \\
t_5 + t_{11} + t_{12} + t_{15} &= p_{12} \\
t_6 + t_{11} + t_{13} + t_{15} &= p_{13} \\
t_7 + t_{12} + t_{13} + t_{15} &= p_{14} \\
t_8 + t_{11} + t_{14} + t_{15} &= p_{23} \\
t_9 + t_{12} + t_{14} + t_{15} &= p_{24} \\
t_{10} + t_{13} + t_{14} + t_{15} &= p_{34}
\end{aligned}$$

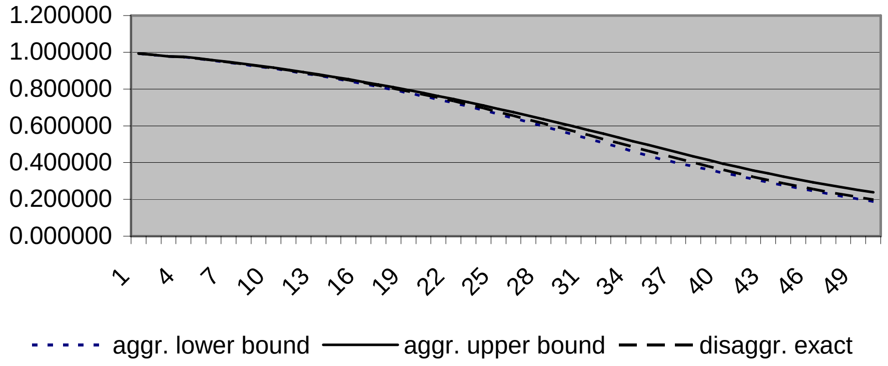
$$t_i \ge 0, \quad i = 0, 1, \ldots, 15.$$

The numerical results for the actuarial present values are given in the following tables and figures (maximal correlation case, minimal correlation case and independent case):

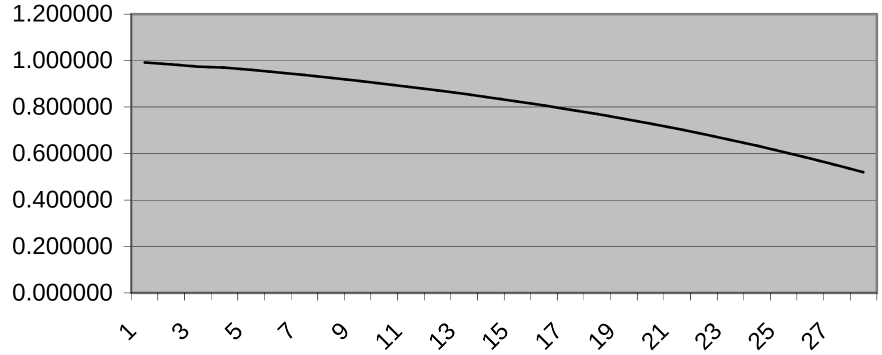
Aggregated LP problems					
T	max. corr.	max. corr.	min. corr.	min. corr.	independent
	lower bound	upper bound	lower bound	upper bound	exact value
1	0.992537	0.992890	0.991832	0.991832	0.991843
2	0.984628	0.985364	0.983158	0.983158	0.983203
3	0.976285	0.977436	0.973983	0.973983	0.974093
4	0.973298	0.974900	0.970094	0.970094	0.970265
5	0.964065	0.966155	0.959883	0.959883	0.960185
6	0.954351	0.956972	0.949107	0.949107	0.949588
7	0.944135	0.947332	0.937739	0.937739	0.938459
8	0.933419	0.937241	0.925774	0.925774	0.926800
9	0.922194	0.926688	0.913207	0.913207	0.914617
10	0.910436	0.915645	0.900019	0.900019	0.901902
11	0.898113	0.904081	0.886176	0.886176	0.888632
12	0.885224	0.891996	0.871681	0.871681	0.874819
13	0.871779	0.879396	0.856544	0.856544	0.860486
14	0.857769	0.866286	0.840734	0.840734	0.845617
15	0.843183	0.852673	0.824202	0.824202	0.830185
16	0.827994	0.838560	0.806863	0.806863	0.814139
17	0.812235	0.824012	0.788680	0.788680	0.797480
18	0.795938	0.809089	0.769634	0.769634	0.780230
19	0.779090	0.793799	0.749672	0.749672	0.762379
20	0.761631	0.778096	0.728703	0.728703	0.743886
21	0.743470	0.761898	0.706616	0.706616	0.724697
22	0.724534	0.745137	0.683328	0.683328	0.704785
23	0.704814	0.727816	0.658810	0.658810	0.684180
24	0.684348	0.709984	0.633074	0.633074	0.662956
25	0.663212	0.691736	0.606164	0.606164	0.641220
26	0.641511	0.673186	0.578162	0.578162	0.619114
27	0.619275	0.654373	0.549078	0.549078	0.596712
28	0.597015	0.635808	0.519429	0.519429	0.574509
29	0.574678	0.617250	n.a.	n.a.	0.552488
30	0.552198	0.597356	n.a.	n.a.	0.530629
31	0.529504	0.577319	n.a.	n.a.	0.508911
32	0.506569	0.557115	n.a.	n.a.	0.487358
33	0.483408	0.536757	n.a.	n.a.	0.466031
34	0.460050	0.516270	n.a.	n.a.	0.444997
35	0.436533	0.495692	n.a.	n.a.	0.424319
36	0.415456	0.475067	n.a.	n.a.	0.404047
37	0.397229	0.454165	n.a.	n.a.	0.383997
38	0.379196	0.433706	n.a.	n.a.	0.364755
39	0.361415	0.413778	n.a.	n.a.	0.346350
40	0.343924	0.394446	n.a.	n.a.	0.328780
41	0.326758	0.375770	n.a.	n.a.	0.312026
42	0.309930	0.357784	n.a.	n.a.	0.296046
43	0.293450	0.340507	n.a.	n.a.	0.280780
44	0.277305	0.323925	n.a.	n.a.	0.266156
45	0.261474	0.308001	n.a.	n.a.	0.252098
46	0.245941	0.292698	n.a.	n.a.	0.238542
47	0.230713	0.278001	n.a.	n.a.	0.225450
48	0.215834	0.263942	n.a.	n.a.	0.212825
49	0.201377	0.250585	n.a.	n.a.	0.200694
50	0.187451	0.238046	n.a.	n.a.	0.189115

Disaggregated LP problems				
T	max. corr. lower bound	max. corr. upper bound	min. corr. lower bound	min. corr. upper bound
1	0.992708	0.992708	0.991832	0.991832
2	0.984993	0.984993	0.983158	0.983158
3	0.976864	0.976864	0.973983	0.973983
4	0.974115	0.974115	0.970094	0.970094
5	0.965142	0.965142	0.959883	0.959883
6	0.955711	0.955711	0.949107	0.949107
7	0.945804	0.945804	0.937739	0.937739
8	0.935424	0.935424	0.925774	0.925774
9	0.924565	0.924565	0.913207	0.913207
10	0.913204	0.913204	0.900019	0.900019
11	0.901309	0.901309	0.886176	0.886176
12	0.888872	0.888872	0.871681	0.871681
13	0.875899	0.875899	0.856544	0.856544
14	0.862378	0.862378	0.840734	0.840734
15	0.848301	0.848301	0.824202	0.824202
16	0.833652	0.833652	0.806863	0.806863
17	0.818467	0.818467	0.788680	0.788680
18	0.802787	0.802787	0.769634	0.769634
19	0.786600	0.786600	0.749672	0.749672
20	0.769850	0.769850	0.728703	0.728703
21	0.752452	0.752452	0.706616	0.706616
22	0.734332	0.734332	0.683328	0.683328
23	0.715486	0.715486	0.658810	0.658810
24	0.695952	0.695952	0.633074	0.633074
25	0.675813	0.675813	0.606164	0.606164
26	0.655180	0.655180	0.578162	0.578162
27	0.634084	0.634084	0.549078	0.549078
28	0.613037	0.613037	0.519429	0.519429
29	0.591992	0.591992	n.a.	n.a.
30	0.570897	0.570897	n.a.	n.a.
31	0.549698	0.549698	n.a.	n.a.
32	0.528379	0.528379	n.a.	n.a.
33	0.506964	0.506964	n.a.	n.a.
34	0.485490	0.485490	n.a.	n.a.
35	0.464009	0.464009	n.a.	n.a.
36	0.442576	0.442576	n.a.	n.a.
37	0.421013	0.421013	n.a.	n.a.
38	0.399953	0.399953	n.a.	n.a.
39	0.379490	0.379490	n.a.	n.a.
40	0.359698	0.359698	n.a.	n.a.
41	0.340639	0.340639	n.a.	n.a.
42	0.322337	0.322337	n.a.	n.a.
43	0.304793	0.304793	n.a.	n.a.
44	0.287969	0.287969	n.a.	n.a.
45	0.271798	0.271798	n.a.	n.a.
46	0.256202	0.256202	n.a.	n.a.
47	0.241113	0.241113	n.a.	n.a.
48	0.226503	0.226503	n.a.	n.a.
49	0.212385	0.212385	n.a.	n.a.
50	0.198832	0.198832	n.a.	n.a.

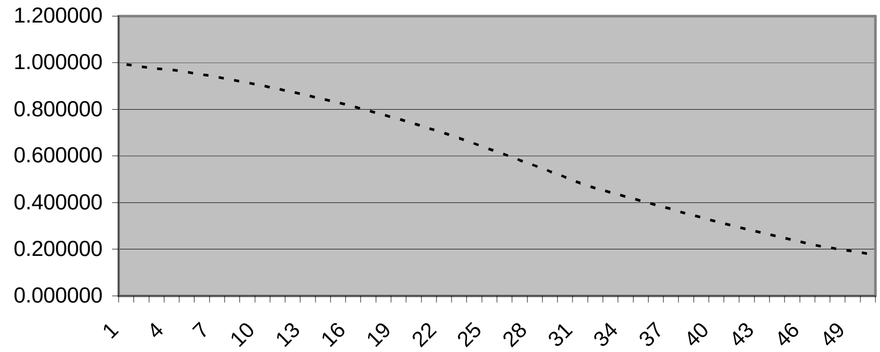
**Maximal pairwise correlation,
aggregated-disaggregated model**



Minimal pairwise correlation



Independent case



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