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# REDUCED MODEL FOR TEMPORAL STABILITY OF A Q-VORTEX

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**ABSTRACT**: This paper presents a numerical investigation upon the temporal stability of a Q type vortex subject to infinitesimal perturbations. The study is based on spectral collocation technique using Chebyshev Gauss-Lobatto points and the eigenvalue problem is obtained in a matrix form and is solved by an Arnoldi type algorithm. Our procedure directly provides relevant information about the state of the fluid system for given parameters, in axi-symmetrical and non-axi-symmetrical mode and also permits a graphical visualization of the perturbed velocity fields.
**KEYWORDS:** temporal stability, trailing vortex, small perturbation.

## 1. INTRODUCTION

We present in this paper a numerical investigation upon temporal stability of a trailing Q type vortex, subject to infinitesimal perturbations. Such problems may be of interest in the field of aerodynamics, where vortices trail on the tip of each wing of the airplane and a stability analysis is needed.

Using a spectral Chebyshev Gauss-Lobatto collocation technique, we developed a numerical procedure which directly provides relevant graphic information about perturbation velocities amplitude for stable or unstable induced modes.

Our paper is organized in the following manner. We set the problem and the perturbations form in Section 2. Section 3 describes in detail the numerical procedure based on collocation technique and relates the results and Section 4 concludes the paper.

## 2. VISCOUS ANALYSIS MODEL FOR TEMPORAL STABILITY

In literature, the properties of swirl flows are presented in a beautiful synthesis in [1] and, later, a study against instability for a trailing vortices class was made in [2].

We consider for our purpose the one-parametric model of the Q-vortex, in form related in [1]

$$U(r)=0\,,\quad V(r)=\frac{Q}{r}\left(1-e^{-r^2}\right),\quad W(r)=0 \tag{1}$$

where $U$, $V$, $W$ are the radial, tangential and axial velocity components, respectively, and $Q$ is the swirl parameter.

As the axial velocity is null, we consider the flow in a base plane, and we perform a stability analysis at the base of the Q-swirl structure. The flow is assumed to be incompressible and the lengths in cylindrical coordinates are non-dimensionalised.

We first obtain the liberalized Navier-Stokes equations, in cylindrical polar coordinates, subject to the base flow of form (1). We perform a linear stability investigation of the proposed base flow, in which the velocity and pressure p fields are decomposed into their mean parts $(U, V, W, P)$ and small perturbations $(u', v', w', p')$. We use in this note the standard form of the perturbations, in detail described in [1]

$$(u', v', w', p')^T = \left(\bar{u}(r), \bar{v}(r), \bar{w}(r), \bar{p}(r)\right)^T e^{i(\alpha z + n\theta - \omega t)} \tag{2}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{p}$ are the disturbances eigenfunctions, $\alpha$ is the axial wavenumber, $n$ is the integer azimuthal wavenumber, $\theta$ is the azimuthal angle and $\omega$ is the complex temporal frequency. We obtained the hydrodynamic temporal stability model of the following form

$$\omega M \bar{s}(r) \equiv -H \bar{s}(r) \quad , \quad \bar{s}(r) = \left(\bar{u}(r), \bar{v}(r), \bar{w}(r), \bar{p}(r)\right)^T \tag{3a}$$

Relation (3a) consists in a set of partial differential equations for the perturbation velocities and pressure, expressed in a matriceal form. The non-zero elements of the 4 × 4 matrix operators *M* and *H* are given by

$$m_{11} = m_{22} = m_{33} = -i$$

and

$$h_{11} = \frac{Vin}{r} - \frac{1}{Re}\left(\Delta + \frac{1}{r^2}\right),\ h_{12} = -\frac{2V}{r} + \frac{2in}{r^2 Re},\ h_{14} = \frac{\partial}{\partial r},$$

$$h_{21} = V' + \frac{V}{r} - \frac{2in}{r^2 Re},\ h_{22} = \frac{Vin}{r} - \frac{1}{Re}\left(\Delta - \frac{1}{r^2}\right),\ h_{24} = \frac{in}{r},$$

$$h_{31} = 0,\ h_{33} = \frac{Vin}{r} - \frac{1}{Re}\Delta,\ h_{34} = i\alpha,$$

$$h_{41} = \frac{\partial}{\partial r} + \frac{1}{r},\ h_{42} = \frac{in}{r},\ h_{43} = i\alpha,$$

where $\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2} - \alpha^2$ and prime denote $V' \equiv dV/dr$. The Reynolds number Re is defined according to the maximum difference of axial velocity and the vortex core radius, as described in [2]. The system (3a) has to be solved in this note subject to the following boundary conditions [7]

$$r = 0$$

$$\text{Case } n = 0 \qquad \bar{u} = \bar{v} = 0,\ d\bar{w}/dr = 0, \tag{3b}$$

$$\text{Case } |n| > 1 \qquad \bar{u} = \bar{v} = \bar{w} = 0; \tag{3c}$$

$$r \to \infty \qquad \bar{u} = \bar{v} = \bar{w} = 0. \tag{3d}$$

In the temporal stability analysis that we will study here, for a given real wavenumber $\alpha$ and given the parameters Re, Q and n, the system (3a-3d) constitutes a linear eigenvalue problem for the complex eigenvalue $\omega$. The flow is considered unstable when the disturbance grows in time, i.e. the imaginary part of the eigenfrequency $\omega$ is positive.

## 3. SPECTRAL COLLOCATION BASED INVESTIGATION

### 3.1. Chebyshev spectral differentiation matrices

Spectral methods are one of the most used technique for the numerical investigations in hydrodynamic stability problems. Many researchers have demonstrated the applicability of this technique with high degree of accuracy, as in [4], [5], [6].

We choose for our study a Chebyshev Gauss-Lobatto collocation approach, for the reasons that Chebyshev polynomials distribute the error evenly, exhibit rapid convergence rates with increasing numbers of terms and cluster the collocation points near the boundaries, diminishing the negative effects of the Runge phenomena [8, 10].

The Chebyshev Gauss-Lobatto points are given explicitly by

$$\xi_j = \cos(\pi j / N),\ j = 0..N \tag{4}$$

In order to compute the derivatives $\partial \bar{s}(r)/\partial r$ and $\partial^2 \bar{s}(r)/\partial r^2$ we constructed the Chebyshev spectral differentiation matrices of first and second order as we describe hereinafter.

Let $\xi_j,\ j=0..N$ be the interpolation points and $\{u_0, u_1, \dots u_N\}$ be the components of the eigenvector $\bar{u}$. We approximate the eigenfunctions with truncated Lagrange series expansions of form

$$\bar{u}(\xi)=\sum_{k=0}^{N} u_k \cdot L_k(\xi),\quad \bar{v}(\xi)=\sum_{k=0}^{N} v_k \cdot L_k(\xi),\quad \bar{w}(\xi)=\sum_{k=0}^{N} w_k \cdot L_k(\xi),\quad \bar{p}(\xi)=\sum_{k=0}^{N} p_k \cdot L_k(\xi).$$

Let as consider, for a simpler explanation, that $N=2$. Then $\xi \in \{1, 0, -1\}$ and the eigenvector $\bar{u}$ can be expressed is the Lagrange expansion of form

$$\bar{u}(\xi)=0.5\cdot\xi(1+\xi)u_0+(1+\xi)(1-\xi)u_1+0.5\cdot\xi(\xi-1)u_2 \tag{5}$$

Differentiating (5) yields

$$\bar{u}(\xi)'=(\xi+0.5)u_0-2\xi\, u_1+(\xi-0.5)u_2 \tag{6}$$

The spectral differentiation matrix $D_2$ is the 3×3 matrix whose j[th] column is obtained by sampling the j[th] term of this expression at $\xi$ = 1, 0, and -1, yielding

$$D_2=(1.5\quad -2\quad 0.5\,;0.5\quad 0\quad -0.5\,;-0.5\quad 2\quad -1.5) \tag{7}$$

For arbitrary N, the entries of Chebyshev spectral differentiation matrix $D_N$ are

$$d_{00}=\frac{2N^2+1}{6},\quad d_{NN}=-\frac{2N^2+1}{6},\quad d_{jj}=\frac{-\xi_j}{2\left(1-\xi_j^2\right)},\quad j=1,\dots,N-1,$$

$$d_{ij}=\frac{c_i}{c_j}\frac{(-1)^{i+j}}{(\xi_i-\xi_j)},\quad i\neq j,\quad i,j=1,\dots,N-1,\quad c_i=\begin{cases}2 & \text{if } i=0,N\\ 1 & \text{otherwise}\end{cases}.$$

For N collocation points, the derivatives are expressed as

$$\bar{u}(\xi)'=D_N\cdot(u_0,u_1,\dots,u_N)^T,\quad \bar{u}(\xi)''=D_N^2\cdot(u_0,u_1,\dots,u_N)^T \tag{8}$$

where $D^2{}_N$ denotes the squared Chebyshev spectral differentiation matrix of first order.

Because Chebyshev polynomials are defined on the interval $\xi\in[-1\quad 1]$ and the physical range of our problem is $r\in[0\quad r_{max}]$, we made use of the Möbius transformation (Figure 1)

$$r(\xi)=a\frac{1-\xi}{b+\xi},\quad b=1+\frac{2a}{r_{max}} \tag{9}$$

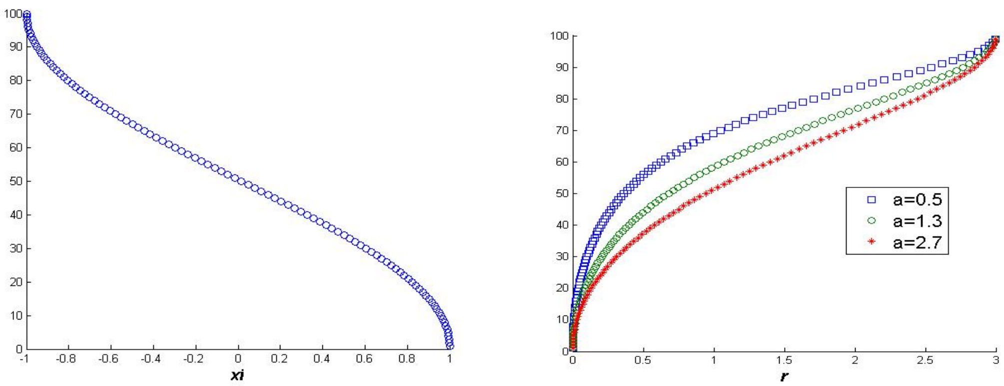



Figure 1. The graphic representation of the Chebyshev collocation points (left, N = 100) and their corresponding points (right), maped into the physical range, for different values of parameter a ($r_{max}$=3)

The differentiation matrices $\Delta_N$ in the physical coordinate r are obtained by a multiplication of the corresponding matrices in the standard interval $[-1\quad 1]$ by the diagonal matrix S with the entries

$$S_{jk}=\left(dr/d\xi|_j\right)^{-1}\delta_{jk}\quad,\quad \Delta_N=S\cdot D_N \tag{10}$$

where $\delta_{jk}$ is the Kronecker delta symbol and $D_N$ is the spectral Chebyshev differentiation matrix in the standard interval.

Because large matrices are involved, we numerically solved (3a-3d) using the Arnoldi type algorithm [8], which provides entire eigenvalue and eigenvector spectrum.

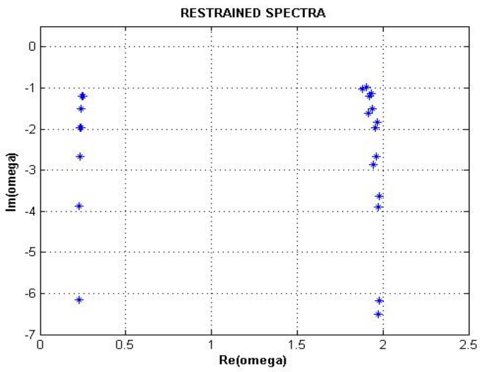
For non-axisymmetrical modes (case $|n| > 1$), our boundary value problem have been solved subject to Dirichlet boundary conditions. This was numerically implemented as part of spectral collocation method by discarding the no effect first and last columns of the Chebyshev differentiation matrices of first and second order and also ignoring the first and last rows.

For implementing the boundary conditions in the axi-symmetrical mode $n = 0$, first we retained the location of rows in matrix *M* and their corresponding columns which were replaced with zero. Then we operate the *H* matrix by replacing some lines with their corresponding lines of identity block matrix, for Dirichlet boundary conditions imposed to $\overline{u}$, $\overline{v}$, $\overline{p}$ (at 0 and $r_{max}$) and $\overline{w}$ (at $r_{max}$), and also, a specific line has been replaced with its corresponding of the block Chebyshev differentiation matrix of first order, for Newman condition fulfilled by $\overline{w}$ at limit 0.

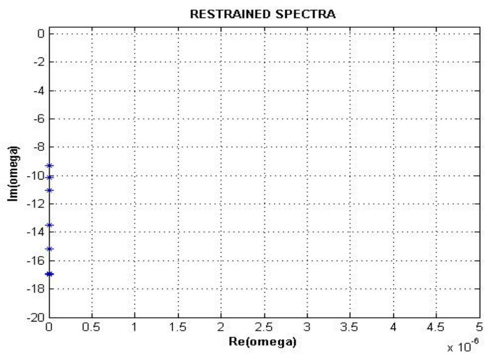
### 3.2. Numerical results

In case n = 2, for given parameters Re = 9000, $\alpha$ = 3.5, Q = 1, N = 90, the fluid system is stable, as can be seen in Figure 2a.

For axy-simmetrical mode n = 0, having the same initial parameters, Re = 9000, $\alpha$ = 3.5, Q = 1, N = 90, the fluid system remains stable (Figure 2b).




a.




b.

Figure 2. a. Spectra obtained for the case n = 2, for given parameters Re = 9000, $\alpha$ = 3.5, Q = 1, N = 90, the fluid system is stable; b. Spectra obtaned for the axysimmetrical mode n = 0, for given parameters Re = 9000, $\alpha$ = 3.5, Q = 1, N = 90, the fluid system is stable

Figures 3 a, b show the eigenvectors for the choosen eigenfrequency $\omega$ = 0.000000000000453 - 9.281409796891595i for the mode n = 2, and $\omega$ = 0.000000000000453 - 9.281409796891595i for the axysimmetrical mode n = 0.

Figures 4 a, b depict the perturbed azimuthal velocity for the considered modes.

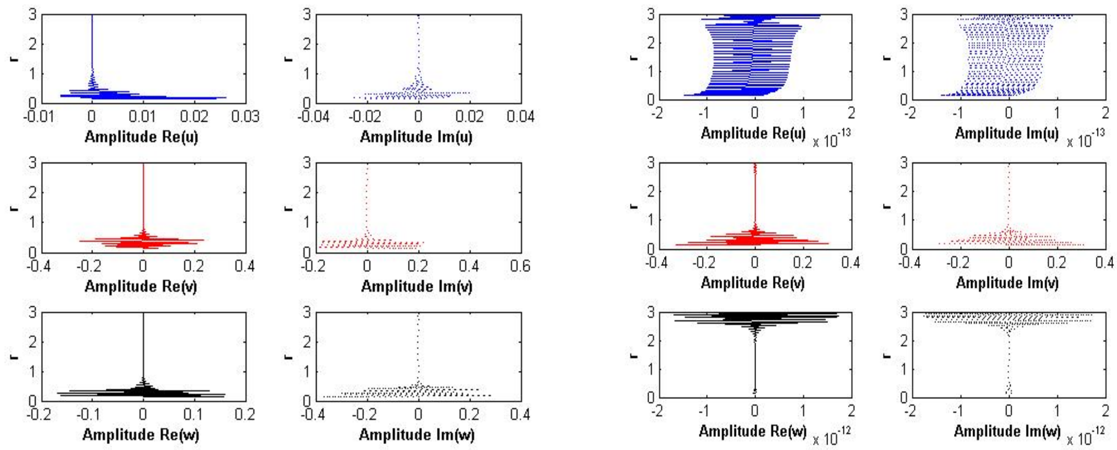


Figure 3. The eigenvectors for n = 2 mode, respectively for n = 0 mode

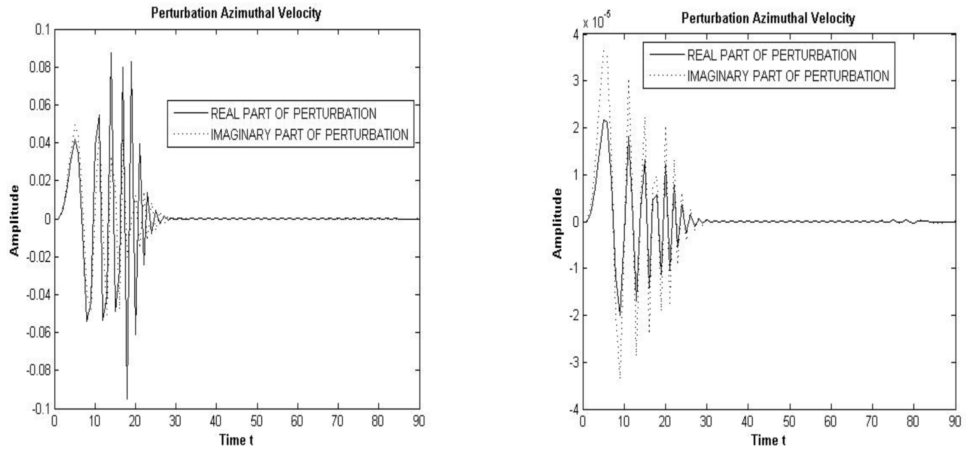



Figure 4. The perturbed azimuthal velocity for n = 2 mode, respectively for n = 0 mode

## 4. SUMMARY AND CONCLUSION

We have made a numerical investigation upon the temporal stability of a trailing swirl flow, namely the Q vortex, subject to infinitesimal perturbations. We have implemented a temporal stability analysis method based on Chebyshev Gauss-Lobatto spectral collocation technique and we numerically obtained information on the state of the fluid system.

Considering the motion in a base plane, we captured the vortex in the next moment of its formation, and we obtained a simplified model, performing a temporal stability analysis at the base of the swirl structure.

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