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ON THE CALCULUS ALGORITHM OF THE FLUID ′S VELOCITY POTENTIALS THROUGH PROFILE GRIDS

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Abstract
Based on the results of [10] and [11], in this paper we present some practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum.
Keywords:
boundary element method, hydrodynamic networks, fluid's velocity potential, Fredholme integral equation, Lagrange interpolation

1. INTRODUCTION

In this paper we present practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum. More precisely, we show the applicability of the boundary element methods (BEM) with real values, and the possibility of solving the integral equation of the velocity potential by using the successive approximation method w.r.t. the parameters _ (fluid's density) and h (thickness variation of fluid stratum), and using the Lagrangian interpolation formula through five points for the calculation of the derivatives of the velocity potential.

The rest of the paper is organized as follows: in section 2 we state the problem and some theoretical considerations that are needed. In section 3 we present the calculus algorithm for the study of the compressible fluid's stationary movement, together with its practical aspects.

Section 4 concludes with some ideas for the future work.

2. PRESENTING THE PROBLEM

The fundamental equations (from the CVBEM method) in the problem of the compressible fluid's movement on a axial-symmetric flow–surface, in variable thickness of stratum, could be ([5], [6], [7]):

$$\overline{w}(z)=\overline{V}_m+\int_{L_0}H(z,\zeta)\overline{w}(\zeta)d\zeta+i\iint_{D_{0^*}^{-}}H(z,\zeta)\widehat{q}(\zeta)d\xi d\eta$$

$$F(z)=\overline{V}_m\cdot z+\Gamma\cdot G(z,\zeta_A)+\int_{L_0}H(z,\zeta)F(\zeta)d\zeta+i\iint_{D_{0^*}^{-}}G(z,\zeta)\widehat{q}(\zeta)d\xi d\eta \qquad (1)$$

where: A – is a fixed point on the base profile L_0; t – is the grid step; Γ – is the circulation around L_0.

$$H(z,\zeta)=\frac{1}{2ti}ctg\frac{\pi}{t}(z-\zeta)$$

$$G(z,\zeta)=\frac{1}{2\pi i}\ln\sin\frac{\pi}{t}(z-\zeta) \qquad (2)$$

$$\widehat{q}(\zeta)=2\frac{\partial\overline{w}}{\partial\overline{\zeta}}=-\left[v_x\frac{\partial\ln p^*}{\partial\xi}-v_y\frac{\partial\ln p^*}{\partial\eta}\right],\qquad p^*=\frac{\varsigma\cdot h}{\varsigma_0}$$

where: ς – is the fluid's density, h – is a function that represents the thickness' variation of the fluid stratum.

D_0^- – bounded simple convex domain, defined as:

$$D_{0*}^- : \left[-\frac{t}{2} \langle \xi \langle \frac{t}{2}, -\left(t+\frac{l}{2}\right) \langle \eta \langle \left(t+\frac{l}{2}\right) \right] \tag{3}$$

where: l – is the projection of L_0 profile's frame on the Oy axis.

Our purpose is to solve the fundamental equations (1) (obtained from the CVBEM method) using (BEM) in real variables. For doing so, we consider the fundamental integral–equation of the complex potential $F(z) = \varphi + i\psi$ and transform it into an integral equation with real variables, i.e. we build the integral equation of the velocity potential φ (s) (ψ(s) is the flow rate function).

Theorem 2.1. *[7], [11] In the subsonic motion of the compressible fluid through the profile grid, on an axial–symmetric flow–surface, in variable thickness of stratum, the velocity potential* φ (s), s ∈ L_0 *is the solution of the integral equation (4):*

$$\varphi(s) + \int_{L_0} \varphi(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b(s) + \iint_{D_{0*}^-} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \tag{4}$$

where:

s (x_0, y_0) *and* σ(ξ,η) *are the curvilinear coordinates of the fixed point A on the* L_0 *base profile;*

$$b(s) = 2\left(x_0 V_{mx} + y_0 V_{my}\right) + \Gamma M(s, \sigma_A) + \int_{L_0} [\psi(s) - \psi(\sigma)] \frac{dN}{d\sigma} d\sigma$$

$$M(z_0, \zeta) = \frac{1}{\pi} arctg \frac{th \frac{\pi}{t}(\eta - y_0)}{tg \frac{\pi}{t}(\xi - x_0)} \tag{5}$$

$$N(z_0, \zeta) = \frac{1}{\pi} \ln \sqrt{\frac{1}{2}\left[ch \frac{2\pi}{t}(\eta - y_0) - \cos \frac{2\pi}{t}(\xi - x_0) \right]}$$

v_{mx}, v_{my} - *are the components of the asymptotic mean velocity* v_m.

Proposition 2.1. *[10],[11] In the case of an axial-subsonic movement of a perfect and compressible fluid through profile grids, the flow rate function is determined from the boundary condition (6):*

$$\psi(s) = u_0 \cdot \int_0^s p^*(s)\left(\frac{R}{R_0}\right) ds, \qquad u_0 = \omega R_0, \tag{6}$$

where:

- ω *is the angular rotation velocity of the profile grid;*
- R_0 *defines the origin of the axis system related to the turbine's axis.*

Equation (4) is an integro–differential equation. In this section, we will show a possibility of solving this equation applying the *method of successive approximation* (the iteration method), using also the result from [8] about the order of the term containing the double integral expression:

$$\varphi_{\hat{q}}(s) = \iint_{D_{0*}^-} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \tag{7}$$

Proposition 2.2. *[10], [9] In the case of the subsonic movement of the compressible fluid through the profile grid on an axial–symmetric flow–surface, in variable thickness of stratum, the integral equation of the velocity potential* $\varphi : D_{0*}^- \to \Re$ *is solvable by applying the method of successive approximations w.r.t. the parameter* $p^* = \frac{\varsigma \cdot h}{\varsigma_0}$.

Proof. For isentropic processes , by the Bernoulli–equation, we obtain:

$$\varsigma = \varsigma_0 \left(1 - \frac{\gamma - 1}{2} \frac{v^2}{c_0^2}\right)^{\frac{1}{\gamma - 1}}, \qquad v^2 = v_\tau^2 + v_n^2, \; v_\tau = \frac{d\varphi}{ds}, \qquad v_n = \frac{1}{p^*}\frac{d\psi}{ds} \tag{8}$$

where:

- γ is the adiabatic constant;
- c_0 is the sound velocity in the zero velocity point;
- v_τ and v_n are, respectively, the tangential and normal velocities on L_0.

In the first approximation it is assumed that $\varsigma = \varsigma_0$ = constant and $p^* = p^{*(0)}$ = constant. Thus, from (2), it results that $q^{(0)}(\sigma) = 0$. Hence, in the integral equation (4) the double integral (7) is neglected and results the following Fredholme integral equation of second type, with continuous nucleus:

$$\varphi^I(s) + \int_{L_0} \varphi^I(s) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b^I(s) \tag{9}$$

From solving equation (9) we obtain φ^I , and furthermore from (6), (8), (12) ψ^I, ς^I are obtained. Finally, using the relation:

$$p^* = \frac{\varsigma \cdot h}{\varsigma_0}, \qquad \hat{q}(\sigma) = -grad\varphi \cdot grad \ln p^* \tag{10}$$

a p^{*I} and $\hat{q}^I(\sigma)$ are determined.

In the second iteration $p^* = p^{*I}$ is assumed and for the determination of $\varphi^{II}(s)$ the following Fredholme integral equation of second type, with continuous nucleus, will be solved:

$$\varphi^{II}(s) + \int_{L_0} \varphi^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b^{II}(s) + \iint_{D_{0^*}^-} q^I(\sigma) N(s,\sigma) d\xi d\eta \tag{11}$$

where a φ^I and $b^{II}(s)$ are previously calculated from (6) and (5), respectively.

From solving equation (11), we obtain φ^{II} . Furthermore, from (6), (8), (12) and (10) ψ^{II}, ς^{II}, p^{*II} and $\hat{q}^{II}(\sigma)$ are obtained, respectively. Next, the third approximation might be done by assuming $p^*=p^{*II}$, and so on.

Proposition 2.3. *[10] Having given the values of the velocity potential on each element of the L_0 profile's division, the tangential velocity v_τ may be calculated in each division element of the L_0 basic profile's boundary by the formula, given by the Lagrange interpolation method through five points:*

$$\begin{aligned} v_{\tau i} &= \frac{d\varphi}{ds}(s_i) = \frac{2}{3\Delta s_i}(\varphi_{i+2} - \varphi_{i-2}) - \frac{1}{12\Delta s_i}(\varphi_{i+4} - \varphi_{i-4}) \\ h &= \Delta s_i = s_{i+1} - s_{i-1}, \\ i &= 1,3,5,\ldots,2n-1, \end{aligned} \tag{12}$$

where n *denotes the number of division elements and by* s_i *we refer to the* i^{th} *element of the division of* L_0.

To ensure the practical functionality of proposition 2.2, i.e. to indicate the solving method of the Fredholme integral equation of second type obtained in each approximation step (equation (6), (11)), let us formulate and prove two more propositions.

Proposition 2.4. *[10], [11] In the first approximation step, solving the velocity potential's Fredholme integral equation of second type is reduced to the solving of four systems of linear algebraic equations.*

Proof. Using the superposition rule of potential streams, we seek the solution of the Fredholme integral equation of second type (9) to be of the form:

$$\varphi^I = \varphi_1^I V_{mx} + \varphi_2^I V_{my} + \varphi_3^I \Gamma + \varphi_4^I u_0, \qquad u_0 = \omega R_o \tag{13}$$

where φ_k^I, $k = 1 \div 4$ are the solutions of the system (14) of integral equations:

$$\begin{aligned} \varphi_1^I(s) + \int_{L_0} \varphi_1^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma &= 2x_0 \\ \varphi_2^I(s) + \int_{L_0} \varphi_2^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma &= 2y_0 \\ \varphi_3^I(s) + \int_{L_0} \varphi_3^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma &= M(s,\sigma_A) \\ \varphi_4^I(s) + \int_{L_0} \varphi_4^I(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma &= b_4(s) \end{aligned} \tag{14}$$

where:
$$b_4(s) = \int_{L_0} \left[\psi^I(s) - \psi^I(\sigma)\right] \frac{dN}{d\sigma} d\sigma \qquad (15)$$

The integral equations (14) could be solved using the Bogoliubov-Krîlov method, conform to which, solving each integral equation reduces to solving a system of linear algebraic equations. Conform to the method, using an arbitrary division, we partition the boundary of L_0 in n subintervals $\Delta s = \Delta\sigma$. Note, that the chosen division might be not uniform, for instance at the trailing or the leading edge, where the variation of the function φ_k^I is stronger from point– to–point, the length of subintervals might be shorter. In each subinterval, the function φ_k^I is assumed to be constant and equal to φ_{kj}^I where j represents the number of the middle–points of the considered subintervals. If the first division–points are debited by even numbers, and the division–points of the middle of the subintervals by odd numbers, then, conform to the approximation method, the integral equations (14) can be approximated by the following systems of linear algebraic equations:
$$\varphi_{ki}^I + \sum_{j=1}^{2n-1} \varphi_{kj}^I \Delta M_{ij} = b_{ki}^I, \qquad i = 1,3,5,\ldots\ldots,2n-1 \qquad k = 1,2,3,4 \qquad (16)$$

where:
$$b_{1i}^I = 2x_i, \qquad b_{2i}^I = 2y_i, \qquad b_{3i}^I = M_{i,A}$$
$$b_{4i}^I = \sum_{j=1}^{2n-1} \Delta\psi_{i,j}^I \left(\frac{dN}{d\sigma}\right)_{i,j} \Delta\sigma_j \qquad (17)$$
$$\Delta\psi_{i,j}^I = \psi_i^I - \psi_j^I, \qquad \Delta\sigma = \sigma_{j+1} - \sigma_{j-1}$$

Solving the algebraic system (16), we obtain φ_{kj}^I in n distinct point from the boundary of L_0. Finally, from equations (13), φ_i^I is determined in each point of the boundary's division.

Proposition 2.5. *[10], [11] In the second approximation step, the Fredholme integral equation (11) of the velocity potential is reduced to solving four systems of linear algebraic equations.*

Proof. From (8) and (10), a ς^I and a $\hat{q}^I(\sigma)$ is determined, respectively. Consequently, using the superposition rule of potential streams, we seek the solution of the Fredholme integral equation of second type (11) to be of the form:
$$\varphi^{II} = \varphi_1^{II} v_{mx} + \varphi_2^{II} v_{my} + \varphi_3^{II}\Gamma + \varphi_4^{II} u_0, \qquad u_0 = \omega R_0 \qquad (18)$$

where φ_k^{II} k = 1 ÷ 4, are the solutions of the system 19 of integral equations:
$$\varphi_1^{II}(s) + \int_{L_0} \varphi_1^{II}(\sigma)\frac{dM(s,\sigma)}{d\sigma}d\sigma = 2x_0 + \iint_{D_{0^*}^-} \hat{q}^I(\sigma)N(s,\sigma)d\xi d\eta$$
$$\varphi_2^{II}(s) + \int_{L_0} \varphi_2^{II}(\sigma)\frac{dM(s,\sigma)}{d\sigma}d\sigma = 2y_0 + \iint_{D_{0^*}^-} \hat{q}^I(\sigma)N(s,\sigma)d\xi d\eta \qquad (19)$$
$$\varphi_3^{II}(s) + \int_{L_0} \varphi_3^{II}(\sigma)\frac{dM(s,\sigma)}{d\sigma}d\sigma = M(s,\sigma_A) + \iint_{D_{0^*}^-} \hat{q}^I(\sigma)N(s,\sigma)d\xi d\eta$$
$$\varphi_4^{II}(s) + \int_{L_0} \varphi_4^{II}(\sigma)\frac{dM(s,\sigma)}{d\sigma}d\sigma = b_4^{II}$$

where:
$$b_4^{II}(s) = \frac{1}{u_0}\int_{L_0}\left[\psi^{II}(s) - \psi^{II}(\sigma)\right]\frac{dN}{d\sigma}d\sigma + \iint_{D_{0^*}^-} q^I(\sigma)N(s,\sigma)d\xi d\eta$$
$$\psi^{II}(s) = u_0\int_0^s \left(\frac{R}{R_0}\right)^2 p^{*I}(s)ds, \qquad p^{*I} = \frac{\varsigma^I \cdot h^I}{\varsigma_0} \qquad (20)$$

Using the numeric method presented in proposition 2.4, by applying the Bogoliubov-Krîlov method, solving (19) is reduced to solving systems of linear algebraic equations.

These systems of linear algebraic equations will have the form:

$$\varphi_{ki}^{II} + \sum_{j=1}^{2n-1} \varphi_{kj}^{II} \Delta M_{ij} = b_{ki}^{I}, \qquad i = 1,3,5,.....,2n-1 \qquad k = 1,2,3,4 \tag{21}$$

where b_{1i}^{II}, b_{2i}^{II}, b_{3i}^{II}, b_{4i}^{II} are obtained by using the Simpson formula for handling the double integral.

Solving the algebraic system (21), we obtain φ_{ki}^{II} in n distinct point from the boundary of L_o. Finally, from equations (18), φ_i^{II}, $i = \overline{1,n}$ is determined in each point of the boundary's division.

3. THE CALCULUS ALGORITHM OF THE FLUID'S VELOCITY POTENTIALS THROUGH PROFILE GRIDS

1. Given are: the entering values into the profile grids of p_1, $v_{1\infty}$, α_1 and the asymptotic mean velocity $\vec{V}_m$; the installation angle λ; the number of profiles n; the density ς_o and the sound velocity c_o corresponding to the null–velocity point. The functions h(σ) and $\frac{R}{R_0}(\sigma)$ are given by their table of values;
2. Conform to the chosen division, the coordinates σ_i (ξ_i, η_i), i = 1, 3, 5, . . . , 2n-1 are determined. The circulation Γ is determined from the Jukovschi-Ciaplâghin condition [10], [11];
3. From equation (5), the values of $\Delta M_{i,j}, \left(\frac{dN}{d\sigma}\right)_{i,j}$, $i, j = 1,3,5,...,2n+1$, are calculated;
4. Using the trapezoid method, ψ_i^I is calculated from the integral equation (6), and, using (17), $b_{ki}^I(k = 1 \div 4)$ are determined;
5. The linear algebraic system (16) is solved, and, thus, φ_{ki}^I is obtained. Furthermore, from (13), φ_i^I is also obtained;
6. Using the Lagrange interpolation formula through five points (12), $v_{\tau i}^I$ is calculated. Next, from (8), v_i^I is determined, and, furthermore, ς_i^I is also obtained;
7. Using ς_i^I and h = const, from the integral (20), by the trapezoid method, a $\psi_i^{II}(\sigma)$ is determined. Using equations (19) and (20), $b_{ki}^{II}(k = 1 \div 4)$ are obtained;
8. The integral equations (19) are solved, transforming them first into a linear algebraic system. Furthermore, φ_{ki}^{II} is obtained, and, from (18), φ_i^{II} is determined;
9. Using the Lagrange interpolation formula through five points (12), $v_{\tau i}^{II}$ is calculated. Next, from (8), v_i^{II} is determined, and, furthermore, ς_i^{II} is also obtained. Furthermore, using ς_i^{II}, the next iteration $\varsigma_i = \varsigma_i^{II}$ can be calculated, h=variable, and the algorithm continues.

4. CONCLUSION AND FURTHERWORK

We have shown some practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial–symmetric flow–surface, in variable thickness of stratum, namely :

- the usage of the boundary element method with real values;

- the applicability of the successive approximation method w.r.t. the parameters ς (fluid's density) and *h* (thickness variation of fluid stratum) for solving the integral equation of the velocity potential;
- the usage of the Lagrangian interpolation formula through five points for calculating the derivatives of the velocity potential.

Regarding practical applicability of our algorithm, our plans for the near future are:

- make more test cases w.r.t. several input (geometrical and hydrodynamical) values of the velocity potentials taken from practical experiments involving profile grids;
- study the possibility of applying the algorithm (i.e. the approximation methods) for the calculation of other fluid–characteristics.

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