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MATHEMATICAL MODEL FOR THE BLOOD FLOW IN CAPILLARY VESSELS

[1.]Sonia PETRILA, [2.]Balazs ALBERT

[1.] "Tiberiu Popoviciu"Informatics Highschool, Cluj-Napoca, ROMANIA
[2.] University "Babes-Bolyai", Cluj-Napoca, ROMANIA

Abstract:
In this paper we develop an original model for the blood flow in capillaries. In the first approach the Stokes system is accepted for the blood flow in capillaries and the fluid is considered to be incompressible. The vessels' walls have a linear elastic and permeable behavior. For the second model a non-Newtonian, rheological model for the blood flow, with a non-constant viscosity coefficient, is used, the walls of the capillaries being linearly elastic, permeable and porous.
Key words:
blood flow in capillary vessels, Newtonian-model, rheological model, elastic permeable porous walls.

1. INTRODUCTION

The most important aspect of the blood flow in capillaries is to supply "with food" the living cells of the organs and to remove the byproducts from every cell. The capillary vessels are built so that molecules with different dimensions can penetrate through the tissues in the surroundings of the capillaries in both ways. Generally capillaries are considered as tubes with very thin and porous walls, through which the transport of certain substances are realized. The presence of these pores and the small diameter of the capillaries distinguish these types of vessels from the others. Due to this reduced diameter and the slow character of the flow we can neglect the non-stationary (pulsating) aspect connected to the rhythmical pumping of the blood by the heart. Furthermore we can neglect the inertial (convective) aspects connected to the viscosity of the blood. Moreover the permeable (porous) character of the capillaries is dominating the elasticity of the vessels' walls.

2. NEWTONIAN MODEL FOR THE BLOOD FLOW IN CAPILLARIES

The first model we propose accepts for the blood flow in thin vessels the Stokes system for incompressible fluids, taking into consideration that the Reynolds number is small. Implicitly it is accepted that the blood is homogenous, the viscosity is constant, the flow has a laminar character and there are no exterior field forces.

The vessels' walls have a linear elastic and permeable behavior and the fluid (substance) change through these walls, very small in volume, respects the Starling hypothesis [7]. This classical hypothesis, which was checked experimentally later by many researchers (Mauro [3], Meschia [4], etc.), maintains the fact that the mass debit through this kind of capillary wall is proportional with the pressure difference between the exterior and the interior of the capillary tube. Moreover, using the results of Beavers and Joseph [1], it is accepted the existence of a slip condition, along the permeable surface, which is "covered" by a porous media, an essential condition confirmed also experimentally.

For simplification we accept the axi-symmetric character of the flow, the axis of symmetry being Oz. Using the cylindrical coordinates (r,θ,z), the motion domain will be, at every time t:

$$\Omega(t) \equiv \{(r,\theta,z)/r < R+\eta(z,t), \quad \theta \in [0,2\pi), z \in (0,L)\}, \tag{1}$$

where R and L are the (initial) radius and the length of the tube respectively, η is the elastic displacement of the wall $\Sigma(t) \equiv \{r = R+\eta(z,t), z \in (0,L)\}$ at the considered moment.

Noting by (u,v) the velocity components of the blood in the directions z and r respectively, by p the pressure while by μ the dynamic viscosity coefficient, the motion equations (Stokes) and the continuity equation becomes

$$\frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2}$$

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial z} = 0 \tag{4}$$

For the boundary conditions, noting by $\overline{p}$ the average of the pressure values given in the respective section, we get:

$$\frac{\partial u}{\partial r} = 0 \text{ and } v = 0 \text{, for } r = 0 \tag{5}$$

$$\frac{\partial u}{\partial r} = -\frac{\beta}{\sqrt{k}}u \text{ and } v = K(p-\nu) \text{, for } r = R + \eta \tag{6}$$

$$\overline{p}(r,z) = p_a \text{, for } z = 0 \tag{7}$$

$$\overline{p}(r,z) = p_v \text{, for } z = L. \tag{8}$$

Here $\frac{\partial u}{\partial r} = -\frac{\beta}{\sqrt{k}}u$ is the Beavers-Joseph slip condition, where β is the slip parameter, while k is the specific permeability of the porous media, $v = K(p-\nu)$ is the consequence of the Starling law, where K is the constant permeability of the wall, while ν (built by the interstitial and osmotic pressure) is a given constant. Concerning to p_a and p_v they are the arterial and venous pressure, both supposed constants.

Reffering to the elasticity of the capillary wall, accepting the linear elastic membrane model, the radial component of the stress can be expressed by the radial displacement η, such that:

$$T_r = \rho_m h\frac{\partial^2 \eta}{\partial t^2} + \frac{hE}{1-\sigma^2}\frac{1}{R^2}\eta + p_{ref}\frac{\eta}{R}, \tag{9}$$

where h is the thickness of the membrane, E the Young modulus, σ the Poisson coefficient, ρ_m is the density of the capillary wall, while p_{ref} is the reference pressure in the ”unperturbed” state, supposed to be constant (the above mentioned p is in fact $p - p_{ref}$).

It is obvious that on a this kind of elastic wall the kinematic condition for the continuity of the pressure, evaluated on the deformed interface $\Sigma(t)$, must be satisfied, namely

$$\frac{\partial \eta(z,t)}{\partial t} = v(R+\eta(z,t),z) \text{ and } u(R+\eta(z,t),z) = 0. \tag{10}$$

These conditions together with the previous Beavers-Joseph and Starling conditions lead to $\frac{\partial \eta}{\partial t} = K(p-\gamma)$ and $\frac{\partial u}{\partial r} = 0$ for $r = R + \eta$ respectively.

Concerning the dynamic condition, it implies the continuity of the stress along the deformable interface (wall). As the constitutive law accepted in this case is that of the Newtonian fluid, we must have along $\Sigma(t)$

$$[(p - p_{ref})[\mathbf{T}] - 2\mu[\mathbf{D}]]\mathbf{n}\cdot\mathbf{e}_r = T_{r,} \tag{11}$$

which leads to

$$[(p - p_{ref})[\mathbf{T}] - 2\mu[\mathbf{D}]]\mathbf{n}\cdot\mathbf{e}_r\left(1+\frac{\eta}{R}\right)\sqrt{1+\left(\frac{\partial \eta}{\partial t}\right)^2} = T_{r,} \tag{12}$$

on $\Sigma(t)$, at any time t.

3. RHEOLOGICAL NON-NEWTONIAN MODEL

In the previous model the blood was investigated as a Newtonian fluid and the system of equations was the Stokes system. Now we accept for the blood a rheological non-Newtonian representation with a non-constant viscosity coefficient. All the other assumptions (non-stationary character, incompressibility, homogeneity, linear elasticity, porosity of the wall) are the same, like in the previous model. The Starling hypothesis and the Beavers-Joseph slip condition are also fulfilled.

We accept again the axi-symmetric character of the blood flow in the capillary tube, the axis of symmetry being Oz. Using the cylindrical coordinates (r,θ,z), the motion domain will be, at every time t, $\Omega(t) \equiv \{(r,\theta,z)/r < R+\eta(z,t),\ \ \theta \in [0,2\pi), z \in (0,L)\}$, where R, L, $\eta(z,t)$ and $\Sigma(t)$ have the same meaning as in the previous model.

In the meridian plane $\theta = const$ if u_z and u_r are the components of the velocity in z and r directions, if p is the pressure (evaluated to a reference pressure p_{ref}), then in the absence of the exterior forces, the mass conservation principle (continuity equation) can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0 . \tag{13}$$

Concerning the flow equations they are obtained from the general Cauchy motion equations, where for the stress tensor we accept the following representation (rheological model for blood)

$$\mathbf{T} = -[p + \lambda(\frac{\partial K}{\partial \dot\gamma}\dot\gamma + \frac{\alpha}{\eta_p}\mathrm{K}^2)]\mathbf{I} + 2(\eta_s + \eta_{RBC})\mathbf{D}, \tag{14}$$

where $\mathbf{D}$ is the rate of strain tensor while $\mathbf{I}$ the unity tensor, p the physical pressure, while η_{RBC} is given by (the Cross model):

$$\eta_{RBC} = \frac{\eta_0^*}{1+(k\dot\gamma)^{1-n}} \equiv \eta_p + \lambda \mathrm{K}(\dot\gamma) . \tag{15}$$

with $\dot\gamma = |4I_2|^{1/2}$, I_2 being the second invariant of the rate of strain tensor $\mathbf{D}$, η_s the plasma viscosity, η_p and η_0^* the viscosity coefficients of the blood, α the "relaxation time", k is a time constant for the shear thining behavior, n the shear thining index, α the mobility parameter, while the function

$$K(\dot\gamma) = \frac{1}{\lambda}(\frac{\eta_0^*}{1+(k\dot\gamma)^{1-n}} - \eta_p),\ \text{for } \lambda > 0 \tag{16}$$

is the so called normal function in the variable $\dot\gamma$, which measures the variation of deformation.

For sake of simplicity we denote

$$\eta(\dot\gamma) = \lambda K(\dot\gamma) + \eta_s + \eta_p \equiv \eta_s + \eta_{RBC} ,$$

$$L = -\frac{2\alpha\lambda}{\eta_p}\frac{\partial \mathrm{K}}{\partial\dot\gamma} - \lambda\frac{\partial^2 \mathrm{K}}{\partial\dot\gamma^2}\ddot\gamma \ \text{ and}$$

$M = -\dfrac{k\eta_0^*(1-n)(k\dot\gamma)^{-n}}{[1+(k\dot\gamma)^{1-n}]^2}$, so that, expressing the tensor $\mathbf{D}$ and the other operators ($\dfrac{\partial}{\partial x_i} = \ldots$etc.) in cylindrical coordinates, we arrive to the following two equations of flow (in u_z and u_r, $u_\theta = 0$)

$$\begin{aligned}\rho(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}) &= -\frac{\partial p}{\partial r} + \eta(\dot\gamma)(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial^2 u_r}{\partial z^2}) + \\ &+ L\frac{\partial\dot\gamma}{\partial r} + M[-\frac{\partial\ddot\gamma}{\partial r} + 2\frac{\partial\dot\gamma}{\partial r}\frac{\partial u_r}{\partial r} + \frac{\partial\dot\gamma}{\partial z}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r})],\end{aligned} \tag{17}$$

$$\begin{aligned}\rho(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}) &= -\frac{\partial p}{\partial z} + \eta(\dot\gamma)(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}) + \\ &+ L\frac{\partial\dot\gamma}{\partial z} + M[-\frac{\partial\ddot\gamma}{\partial z} + 2\frac{\partial\dot\gamma}{\partial z}\frac{\partial u_z}{\partial r} + \frac{\partial\dot\gamma}{\partial r}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r})].\end{aligned} \tag{18}$$

These evolution systems are completed by the boundary conditions which express both the presence of a pressure gradient along the *Oz* axis (in accord with the rhythmical pumping of the blood in vessels) and the elastic character of the permeable, porous wall, more precisely

$$\frac{\partial u_z}{\partial r} = 0 \text{ and } u_r = 0 \text{ for } r = 0, \tag{19}$$

$$\frac{\partial u_z}{\partial r} = -\frac{\beta}{\sqrt{\mathrm{K}}} u_z \text{ and } u_r = \mathrm{K}(p - \nu) \text{ for } r = R + \eta(z,t), \tag{20}$$

(the first relation in (20) expresses the Beavers-Joseph slip condition with the slip parameter β while *K* is the specific permeability of the porous media, meantime $u_r = \mathrm{K}(p-\nu)$ is the consequence of the Starling law, with K the constant permeability of the wall, ν ,built by the interstitial and osmotic pressure, supposed to be fixed and *a* a given constant), while for the pressure we have

$$p = \frac{\cos(\omega t)}{a} + p_m \text{ for } z = 0, \text{ where } a > 0, \tag{21}$$

$$p = \frac{\cos(\omega t)}{a + L} + p_m \text{ for } z = L, \text{ where } a > 0, \tag{22}$$

where $p_m = \dfrac{\int_0^R f(r)dr}{R} \equiv f(\xi)$, f is a primitivable and derivable function according to *r*, with a maximum for $r = 0$ and a minimum for $r = R + \eta$. It can be remarked that $p\big|_{z=0} > p\big|_{z=L}$ at any time of the motion (0, *T*).

Observations: These boundary conditions on the "edges" $z = 0$ and $z = L$ of the capillary are in accord with the acceptance of a representation for the pressure of the type $p = \dfrac{\cos(\omega t)}{a+z} + f(r)$, namely of a pressure gradient (in the cylindrical reference $\vec{e}_z, \vec{e}_r$) under the form $\mathrm{grad}p = -\dfrac{\cos(\omega t)}{(a+z)^2}\vec{e}_z + f'(r)\vec{e}_r$. If $f'(0) = 0$ and $f'(R + \eta(z,t)) = 0$ we have $\mathrm{grad}p\big|_{Oz} = -\dfrac{\cos(\omega t)}{(a+z)^2} = \mathrm{grad}p\big|_{r=R+\eta(z,t)}$, in accord with the motion of the pressure in the interior of the capillary.

On the other hand accepting for the capillary wall the linear elastic membrane model, the radial component of the membrane's stress is expressed by the radial displacement η as follows

$$T_r = \rho_m h \frac{\partial^2 \eta}{\partial t^2} + \frac{hE}{1-\sigma^2}\frac{1}{R^2}\eta + p_{ref}\frac{\eta}{R}, \tag{23}$$

where *h* is the thickness of the membrane, *E* the Young modulus, σ the Poisson coefficient, ρ_m is the density of the capillary wall while p_{ref} is the reference pressure in the "unperturbed" state. It is evident that this stress must coincide with the stress generated by the blood on the same radial direction, namely $\vec{T} = \mathbf{T}\vec{n} \cdot \vec{e}_r = T_r$, which represents the relation for determining *f* (the pressure) or $\eta(z,t)$.

At the same time the kinetic condition must be satisfied on the elastic wall, $\dfrac{\partial \eta}{\partial t} = u_r(R + \eta(z,t), z)$ but also $u_z(R + \eta(z,t), z) = 0$, what leads to $\dfrac{\partial \eta}{\partial t} = \mathrm{K}(p(z,r,t) - \nu)$ and $\dfrac{\partial u_z}{\partial r} = 0$ for $r = R + \eta$ respectively.

It can be remarked that the last relation, together with $u_z(R + \eta(z,t), z) = 0$, implies $u_z = 0$ in the whole a surrounding of the elastic wall while the conditions $\dfrac{\partial u_z}{\partial r} = 0$ and $u_r = 0$ for the axis $r = 0$ show that $\vec{u} = u_z \vec{e}_z$ depends only on *z* and *t* so that we have a pulsating flow along the axis *Oz*, which "calms down" on the elastic wall ($u_z = 0$) where the exterior imposed pressure will have a minimum. At the same time from $\frac{\partial \eta}{\partial t} = \mathrm{K}(p(z,r,t) - \nu)\Big|_{r=R+\eta(z,t)}$ we obtain $\frac{\partial^2 \eta}{\partial t^2} = \mathrm{K}(\frac{-\omega \sin(\omega t)}{a+z})\Big|_{r=R+\eta(z,t)}$. This last evaluation for $\frac{\partial^2 \eta}{\partial t^2}$ permits us to make precise the condition on the capillary wall (linear elastic membrane), namely the expression of the "equilibrium" condition $\mathbf{T}\vec{n}\cdot\vec{e}_r = T_r$ in cylindrical coordinates. More precisely if we note by

$$P = [p + \lambda(\frac{\partial \mathsf{K}}{\partial \dot{\gamma}}\ddot{\gamma} + \frac{\alpha}{\eta_p}\mathsf{K}^2)] \tag{24}$$

the equilibrium condition becomes

$$-\frac{P}{\sqrt{1+(\frac{\partial \eta}{\partial z})^2}} + \frac{2\eta(\dot{\gamma})}{\sqrt{1+(\frac{\partial \eta}{\partial z})^2}}\frac{\partial u_r}{\partial r} - \frac{\frac{\partial \eta}{\partial z}}{\sqrt{1+(\frac{\partial \eta}{\partial z})^2}}(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}) = \rho_m h(-\frac{\mathrm{K}\omega \sin(\omega t)}{a+z}) + \frac{hE}{1-\sigma^2}\frac{1}{R^2}\eta + p_{ref}\frac{\eta}{R}, \tag{25}$$

what provides an equation to determine the deformation of the capillary wall, namely $\eta(z,t)$, so that the whole set of unknowns of our problem can be determined.

4. CONCLUSIONS

In this paper we elaborated an original mathematical model for the blood flow in capillary vessels. First we presented a model where the blood was accepted as a Newtonian fluid. In the second approach we extended the model to a more general rheological (non-Newtonian) blood behavior which stands closer to the realistic phenomena.

The previous model will be approached numerically in another paper where we will also consider a more general behavior for the blood.

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