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A THREE DIMENSIONAL AXI-SYMMETRIC MODEL FOR THE BLOOD FLOW IN THIN VESSELS

[1.] Sonia PETRILA, [2.] Balazs ALBERT

[1.] "Tiberiu Popoviciu" Informatics Highschool, Cluj-Napoca, ROMANIA
[2.] University "Babes-Bolyai", Cluj-Napoca, ROMANIA

Abstract:
In this paper, taking into consideration the rheological Cross model, we elaborate a three dimensional axi-symmetric numerical model for the blood flow in thin vessels with adequate algorithms. First we consider the vessel wall to be rigid than we take into account the elastic and porous behavior of this wall what leads to a more realistic approach of the problem.
Key words:
rheological Cross model, blood flow in thin vessels, elastic porous walls.

1. INTRODUCTION

In the proposed model, set up in this work, we accept for the blood flow a rheological non-Newtonian representation with a non-constant viscosity coefficient. The blood is considered to be a homogenous fluid, the flow has a laminar character and there are no exterior field forces. The walls of the vessels have a linear elastic, permeable and porous behavior. For the blood flow in the vessel we also accept an axial symmetry (Oz being the axis of symmetry). Using the cylindrical coordinates (r,θ,z), the motion domain will be, at every time t, $\Omega(t) \equiv \{(r,\theta,z)/r < R+\mu(z,t),\ \ \theta \in [0,2\pi), z \in (0,L)\}$, where R and L are the (initial) radius and the length of the tube(vessel) respectively, μ is the elastic displacement of the wall $\Sigma(t) \equiv \{r = R+\mu(z,t), z \in (0,L)\}$ at the considered moment.

2. NUMERICAL MODEL FOR THE BLOOD FLOW IN THIN VESSELS

This model is based on the rheological Cross model, where the viscosity coefficient for the blood is not constant and can be written as follows

$$\eta(\dot{\gamma}) = \eta_s + \frac{\eta_0^*}{1+k\dot{\gamma}^{1-n}}\ , \tag{1}$$

where η_s is the plasma viscosity, $\dot{\gamma} = |4I_2|^{1/2}$, I_2 being the second invariant of the rate of strain tensor **D** (**D** is the rate of strain tensor), η_0^* the viscosity coefficient of the blood and k is a time constant for the shear thining behavior.

In the absence of the exterior forces we can write for the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0\ , \tag{2}$$

where u_z and u_r are the components of the velocity in z and r directions respectively. For the flow equations we use the general Cauchy motion equations, where we accept for the stress tensor the following representation

$$\mathbf{T} = -[p + \lambda(\frac{\partial K}{\partial \dot{\gamma}}\ddot{\gamma} + \frac{\alpha}{\eta_p}K^2)]\mathbf{I} + 2(\eta_s + \eta_{RBC})\mathbf{D},. \tag{3}$$

where where **D** is the rate of strain tensor while **I** is the unity tensor, p the physical pressure and η_{RBC} is given by the Cross model

$$\eta_{RBC} = \frac{\eta_0^*}{1+(k\dot{\gamma})^{1-n}}, \tag{4}$$

with n the shear thining index, α the mobility parameter, while the function

$$K(\dot{\gamma}) = \frac{1}{\lambda}\left(\frac{\eta_0^*}{1+(k\dot{\gamma})^{1-n}} - \eta_p\right), \text{ for } \lambda > 0, \tag{5}$$

is the so called normal function of the variable $\dot{\gamma}$, which measures the variation of deformation.

Using the above mentioned Cross model, the continuity equation and the Cauchy motion equations, expressing the tensor **D** and the other involved operators in cylindrical coordinates, we arrive to the following three dimensional axi-symmetric system

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial z} = 0 \tag{6}$$

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial r} &= \eta(\dot{\gamma})\Delta' u + \\ + L\frac{\partial \dot{\gamma}}{\partial r} + M&\left[-\frac{\partial \ddot{\gamma}}{\partial r} + 2\frac{\partial \dot{\gamma}}{\partial r}\frac{\partial u}{\partial r} + \frac{\partial \dot{\gamma}}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\right]\end{aligned} \tag{7}$$

$$\begin{aligned}\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z}\right) + \frac{\partial p}{\partial z} &= \eta(\dot{\gamma})\Delta' v + \\ + L\frac{\partial \dot{\gamma}}{\partial z} + M&\left[-\frac{\partial \ddot{\gamma}}{\partial z} + 2\frac{\partial \dot{\gamma}}{\partial z}\frac{\partial v}{\partial z} + \frac{\partial \dot{\gamma}}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\right].\end{aligned} \tag{8}$$

where u is the radial velocity (in direction r), v the axial velocity (in direction z) and p is the pressure, while

$$\begin{aligned}\dot{\gamma} &= \sqrt{2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2} \\ L &= -\frac{2\lambda\alpha}{\eta_p}\mathrm{K}\frac{\partial \mathrm{K}}{\partial \dot{\gamma}} - \lambda\frac{\partial^2 \mathrm{K}}{\partial \dot{\gamma}^2}\ddot{\gamma} \\ M &= \lambda\frac{\partial \mathrm{K}}{\partial \dot{\gamma}},\ \mathrm{K}(\dot{\gamma}) = \frac{1}{\lambda}\left(\frac{\eta_0^*}{1+k\dot{\gamma}^{1-n}} - \eta_p\right)\end{aligned} \tag{9}$$

These evolution systems are completed by the boundary conditions which express both the presence of a pressure gradient along the Oz axis (in accord with the rhythmical pumping of the blood in vessels) and the elastic character of the permeable, porous wall

$$\frac{\partial u_z}{\partial r} = 0 \text{ and } u_r = 0 \text{ for } r = 0, \tag{10}$$

$$\frac{\partial u_z}{\partial r} = -\frac{\beta}{\sqrt{\mathrm{K}}}u_z \text{ and } u_r = \mathrm{K}(p - \nu) \text{ for } r = R + \mu(z,t), \tag{11}$$

In the first relation of (11) the Beavers-Joseph slip condition is expressed with the slip parameter β while K is the specific permeability of the porous media, meantime $u_r = \mathrm{K}(p-\nu)$ is the consequence of the Starling law, with K the constant permeability of the wall, ν is built by the interstitial and osmotic pressure, supposed to be fixed, and a is a given constant.

We have calculated the case of the rigid wall and numerical tests have been effectuated for a vessel with radius 100μm. It was used the finite difference method (to simplify the program on a right angle domain), with separate nodes for u, v, p.

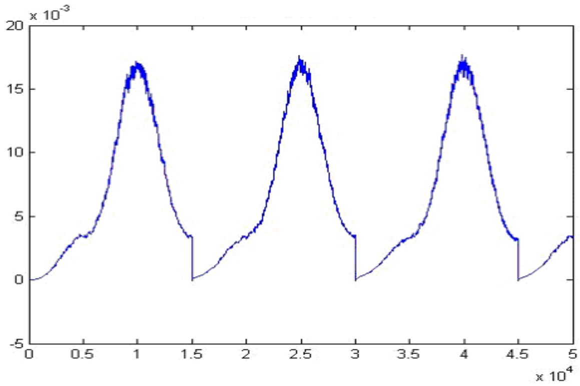

Figure 1. Evolution of the central axial velocity

The evolution in time was made with the Adams-Basforth-Crank-Nicholson method with implicit part for the Laplacian and explicit part for the non-linear components. Thus, at every time step a huge linear system is to be solved but with a sparse matrix. It is necessary to work with small time step (*dt*=0.0001) for assuring the numerical stability. The algorithm presents some oscillations of the calculated solution but these don't disturb the stability. The work efficiency is acceptable (15000 time steps in about 2 hours). The attached figure 1 presents the evolution of the longitudinal velocity in the centre of the vessel for oscillating input and output pressure.

$$pin = 10100\ Pa * fact$$

$$pout = 10000\ Pa * fact$$

$$fact = \begin{cases} \sin \pi t, 0 < t < \frac{1}{2} \\ \frac{3}{2} - \frac{1}{2}\cos(2\pi(t - \frac{1}{2})), \frac{1}{2} < t < \frac{3}{2} \end{cases} \qquad (12)$$

3. NUMERICAL MODEL FOR THE ELASTIC WALLS

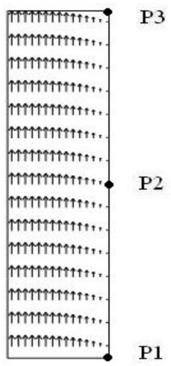

Figure 2. The velocity field at time t=1

One step further we use the presented mathematical model in the case when the elasticity and the porosity of the vessel wall are also considered. The data used for the numerical algorithms were achieved experimentally by C. Balan [1].

We have worked with a small vessel segment of radius $10^{-4} m$ and length of $4*10^{-4} m$. The blood parameters of the proposed model are: $\eta_S = 10^{-3} Pas$, $\eta_0^* = 1 Pas$, $k = 100$, $n = 0.2$, $\lambda = 100$, $\alpha = 50$, the mass density is $1060 kg/m^3$. We considered an oscillatory pressure $pin = 10100 + 150\cos(2\pi t) Pas$ on the input boundary $(z = 0)$ and a constant pressure on the output boundary $pout = 10000 Pa$. The permeability constant is $K = 5*10^{-5}$ and the osmotic pressure is $\nu = 9900$. On the axis of symmetry we imposed the axial symmetry requirements and on the tube walls the Beavers-Joseph condition [2] and the permeability condition (Starling law [5]) $u_r = \mathrm{K}(p - \nu)$.

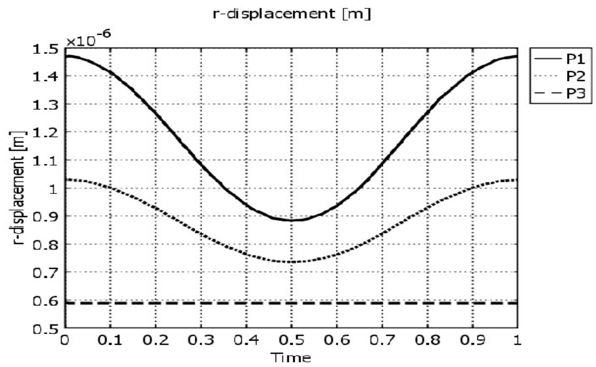

Figure 3. The r-displacement of the wall at points P1, P2 and P3

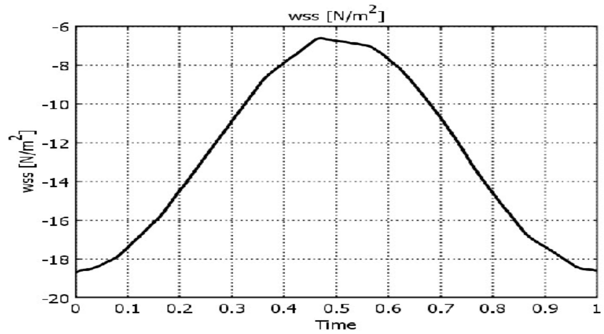

Figure 4. The WSS at point P2

On the attached figures some results of the numerical experiments are presented: on figure 2 the velocity field at the moment $t = 1$, on figure 3 the r-displacement (in the direction of the radius) of the wall at points P1, P2 and P3, on figure 4 the wall shear stress ($WSS = \eta(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r})$ evaluated at P2 is shown.

4. CONCLUSIONS

In this paper by considering a more general rheological model for the blood flow in thin vessels, with adequate numerical algorithms and solvers, we got some results for the longitudinal velocity in the case of the rigid vessel wall. Considering the elastic, porous behavior of the wall we obtained other results for the velocity field, for the r-displacement of the wall and for the wall shear stress (WSS).

In the future we will try to elaborate other numerical algorithms for reducing the calculation time and to eliminate completely the oscillating behavior. We also mention that this implemented rheological model can be easily set up, adjusting the parameters to obtain a better correspondence with the physical measurements.

The completed model can be applied for stenotic arteries, even in the 3-D case (axi-symmetric).

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