ANNALS of Faculty Engineering Hunedoara – International Journal of Engineering
ISSN: 1584 - 2673

ANALYTICAL SOLUTION OF PLANE HARMONIC WAVES IN THIN ORTHOTROPIC ELASTIC PLATES

K. L. VERMA

Department of Mathematics, Government Post-Graduate College Hamirpur, INDIA

ABSTRACT
Analysis for the propagation of plane harmonic waves in an infinite homogeneous orthotropic plate of finite thickness times is studied. The frequency equations corresponding to the symmetric (extensional) and antisymmetric (flexural) wave modes of vibration are obtained and discussed. Special cases of the frequency equations are also discussed. Numerical solution of the frequency equations for transversely isotropic plate is carried out, and the dispersion curves for the first few lower modes are presented for a transversely isotropic plate. The wave motions namely, longitudinal and transverse of the medium are found dispersive and coupled with each other due to the anisotropic effects. The phase velocity of the waves is getting modified due to the anisotropic effects. Relevant results of previous investigations are deduced as special cases.
KEYWORDS: Orthotropic, symmetric, antisymmetric, frequency equations, dispersion

1. INTRODUCTION

Elastic waves are appropriate to measure elastic properties as well as flaws of solid specimens and have received much interest; such as in the use of elastic waves in nondestructive evaluation of concrete structures, laser generated ultrasonic waves, the determination of anisotropic elastic constants of composite materials, recovery of the bonding properties and/or thickness of bonded structures. The proposed high-speed civil transport and advanced aerospace composite materials are required to respond well to the mechanical loading. The materials used in the design of such structures (future supersonic and hypersonic flight and launch vehicles), where high strength to weight ratios in structural components is an important factor, are likely to experience enormous challenges during their flight missions, so they need to perform well under these strenuous conditions for thousands of flight hours. Thus, the increasing use of such advanced high strength, high modulus composites materials, laminated and fiber-reinforced structures must go careful inspection to sort out manufacturing errors and in-service degradation and defect formation. Among the various techniques available, Lamb waves offer a convenient method of evaluating these composite materials**.** Study of wave propagation and vibration in infinite plates, half-spaces and laminates is an area of considerable recent research activity as these structural components are subjected to various transient stresses conditions. The aspect of wave phenomenon is, also of great importance in such materials and study of their behavior will prove to be a good contribution and useful in practical applications.

Based upon Rayleigh's Mathematical formulation, Lamb [1] extensively investigated the properties of these waves and today they are known as Rayleigh-Lamb waves. Compared to the extensive literature on the elastic waves in infinite anisotropic media; relatively little attention has been given to elastic waves in anisotropic plates. Although a entire review of the extensive literature on this subject cannot be undertaken, a number of salient contributions should be mentioned. Propagation of elastic waves in anisotropic homogeneous plate has been studied in detail by authors [2-8]. These studies provide an interesting picture of the rich dispersion characteristic of these guided waves. Several others authors [9-13] have studied free Lamb waves. Wave propagation in anisotropic composite media has also been studied by Nayfeh and Chementi [14] and Yan Li & R. B Thomson [15].

In this article, problem of plane harmonic thermoelastic waves in an infinite homogeneous orthotropic plate of finite thickness is studied. The frequency equations corresponding to the symmetric (extensional) and antisymmetric (flexural) wave modes of vibration are obtained and discussed. Special cases of the frequency equations are also discussed. Numerical solution of the frequency equations is carried out for a transversely isotropic plate, and the dispersion curves are presented. The motions namely, longitudinal and transverse of the medium are found dispersive and coupled with each other due to the anisotropic effects. The phase velocity of the waves is get modified due to the anisotropic effects .Relevant results of previous investigations are deduced as special cases.

2. FORMULATION

Consider a set of Cartesian coordinate system $x_i = (x_1, x_2, x_3)$ in such a manner that x_3- axis is normal to the layering. The basic field equations of motion for an infinite generally anisotropic medium are

$$\sigma_{ij,j} = \rho \ddot{u}_{,i} \tag{2.1}$$

$$\sigma_{ij} = c_{ijkl} e_{kl}, \quad i, j, k, l = 1,2,3, \tag{2.2}$$

where ρ is the density, t is the time, u_i is the displacement in the x_i direction, σ_{ij} and e_{ij} are the stress and strain tensor respectively; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$c_{ijkl} = c_{klij} = c_{ijlk} = c_{jikl} \tag{2.3}$$

Strain-displacement relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2 . \tag{2.4}$$

The stresses and displacements at the surface of the plate are:

$$S(x_3) = (\bar{\sigma}_{13}, \bar{\sigma}_{23}, \bar{\sigma}_{33}), \tag{2.5}$$

$$D(x_3) = (\bar{u}_1, \bar{u}_2, \bar{u}_3), \tag{2.6}$$

and the bar means the amplitudes of the displacement; temperature, stress and the temperature gradient are the function of x_3 only.

The boundary conditions on the plate surfaces are:

$$S(-d) = \mathbf{0}, \tag{2.7}$$

$$S(d) = \mathbf{0}, \tag{2.8}$$

where $\mathbf{0}$ is a zero vector.

When specializing the equations (2.1)-(2.4) for orthotropic the governing equations are

$$(c_{11}u_{1,11} + c_{66}u_{1,22} + c_{55}u_{1,33}) + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} = \rho \ddot{u}_1 \tag{2.9}$$

$$(c_{12} + c_{66})u_{1,12} + (c_{66}u_{2,11} + c_{22}u_{2,22} + c_{44}u_{2,33}) + (c_{23} + c_{44})u_{3,23} = \rho \ddot{u}_2 \tag{2.10}$$

$$(c_{13} + c_{55})u_{1,13} + (c_{23} + c_{44})u_{2,23} + c_{55}u_{3,11} + c_{44}u_{3,22} + c_{33}u_{3,33} = \rho \ddot{u}_3 \tag{2.11}$$

3. ANALYSIS

Having identified the plane of incidence to be the $x_1 - x_3$ plane, then the solution for displacements and temperature for an angle of incidence θ, is proposed:

$$u_j = U_j \exp[i\xi(\sin\theta\, x_1 + \alpha x_3 - ct)], \; i = \sqrt{-1}, \; j = 1, 2, 3 \tag{3.1}$$

where ξ is the wave number, c is the phase velocity (= ω / ξ), ω is the circular frequency, α is still an unknown parameters, U_j are the constants related to the amplitudes of displacement u_1,, u_2, u_3 . Although solutions (3.1) are explicitly independent of x_2, an implicit dependence is contained in the transformation and the transverse displacement component u_2 is non-vanishing in equation (3.1).
Substituting (3.1) in equations (2.9)-(2.11) leads to the coupled equations, the choice of solutions leads to four coupled equations

$$M_{mn}(\alpha)U_n = \mathbf{0} \qquad m, n = 1, 2, 3 \tag{3.2}$$

where

$$M_{11} = F_{11} + c_2\alpha^2, \; M_{13} = F_{13}\alpha, \; M_{22} = F_{22} + c_6\alpha^2, \; M_{33} = F_{33} + c_1\alpha^2 \tag{3.3}$$

$$F_{11} = \sin^2\theta - \zeta^2,\ F_{13} = c_7\sin\theta,\ F_{22} = c_3\sin^2\theta - \zeta^2,\ F_{33} = c_2\sin^2\theta - \zeta^2, \tag{3.4}$$

$$c_1 = \frac{c_{33}}{c_{11}},\ c_2 = \frac{c_{55}}{c_{11}},\ c_3 = \frac{c_{66}}{c_{11}},\ c_6 = \frac{c_{44}}{c_{11}},\ c_7 = \frac{c_{13}+c_{55}}{c_{11}},\ \zeta^2 = \frac{c^2\rho}{c_{11}}$$

The system of equations (3.2) has a non-trivial solution if the determinant of the coefficients of U_1, U_2 and U_3 vanishes, which yields an algebraic equation relating α to c. We obtain a biquadratic polynomial equation in α, which can be written as

$$c_1c_2\alpha^4 + A_1\alpha^2 + A_2 = 0, \tag{3.5}$$

and

$$c_3 + c_6\alpha^2 - \zeta^2 = 0, \tag{3.6}$$

where

$$A_1 = F_{11}c_1 + c_2F_{33} - F_{13}{}^2$$
$$A_2 = F_{11}F_{33} \tag{3.8}$$

Notice that roots of (3.6) correspond to the SH motion, gives a purely transverse wave, which is not affected by the temperature. This wave propagates without dispersion or damping. Equation (3.6) corresponds to the sagittal plane waves, and for the motion in this plane, each α_l (l = 1, 2 ,3, 4) the displacements, stresses amplitudes are

$$q_{3(l)} = -\frac{(F_{11} + c_2\alpha_l^2)}{F_{13}\alpha_l} \tag{3.9}$$

$$r_{33(l)} = [i\xi\{(c_7 - c_2)\sin\theta + c_1\alpha_l q_{3(l)}\}, \tag{3.10}$$

$$r_{13(l)} = i\xi[c_2(\alpha_l + q_{3(l)}\sin\theta)], \tag{3.11}$$

For the SH type wave, one now has

$$r_{23(8)} = -r_{23(7)} = c_6\alpha_7. \tag{3.12}$$

As the equation (3.5) admits solutions for α, having the properties $\alpha_{2l} = -\alpha_{2l-1}$, ($l$ = 1, 2) incorporating this property into Eqs. (3.9) we have.

$$q_{3(2l)} = -q_{3(2l-1)}, \tag{3.13}$$

4. DISPERSION RELATIONS

If the roots of quadratic equation (3.5) are denoted by α_1^2, and α_2^2 then solutions of u_1, and u_3 are then being obtainable as linear combinations of four linear independent solutions corresponding to, α_l (l = 1, 2 ....4) with property $\alpha_{2l-1} = -\alpha_l$, ($l$ = 1, 2). The equations of motion may be used to establish the formal solution for the displacement as

$$(u_1, u_3) = \sum_{l=1}^{4}(1, q_{3(l)})A_l\exp(\iota\xi\alpha_l x_3)\exp[\iota\xi(x_1\sin(\theta) - ct)], \tag{4.1}$$

Using the properties $\alpha_{2l-1} = -\alpha_l$, $l = 1, 2$, we have

$$(u_1, u_3) = (\bar{u}_1, \bar{u}_3)\exp[\iota\xi(x_1\sin(\theta) - ct)] \tag{4.2}$$

where

$$\bar{u}_1 = \sum_{l=1}^{2}(U^{(2l-1)}E_l^+ + U^{(2l)}E_l^-) \tag{4.3}$$

$$\bar{u}_3 = \sum_{l=1}^{2}q_{3(l)}(U^{(2l-1)}E_l^+ - U^{(2l)}E_l^-) \tag{4.4}$$

where in

$$E_l^+ = e^{i\xi\alpha_l d} \text{ and } E_l^{-1} = e^{-i\xi\alpha_l d}\ (l = 1, 2), \tag{4.5}$$

and $U^{(i)}$, (i = 1, 2, 3, 4) are disposal constants. The disposal constants for $U^{(i)}$, are not independent as they are linked through the equations of motion and heat conduction. Here $q_{3(l)}$ are the displacements ratios defined in (3.9).

Combining Eqs.(4.1)- (4.4) and (3.10) - (3.11) with the stress-strain relations and using superposition, we write stresses as

$$(\sigma_{33}, \sigma_{13}) = (\bar{\sigma}_{33}, \bar{\sigma}_{13}) \exp[\iota\xi(x_1 + \alpha x_2 - ct)], \tag{4.6}$$

with

$$\bar{\sigma}_{33} = \sum_{l=1}^{2} r_{33(l)} (U^{(2l-1)} E_l^+ + U^{(2l)} E_l^-) \tag{4.7}$$

$$\bar{\sigma}_{13} = \sum_{l=1}^{2} r_{13(l)} (U^{(2l-1)} E_l^+ + U^{(2l)} E_l^-) \tag{4.8}$$

where $r_{33(l)}$, and $r_{13(l)}$, l= 1, 2, 3, 4. are defined in (3.10)-(3.11). Incorporating this property $\alpha_{2l-1} = -\alpha_l$, into equations (3.10) and (3.11), we conclude the further restrictions.

$$r_{33(2l)} = r_{33(2l-1)}, \tag{4.12}$$

$$r_{13(2l)} = -r_{13(2l-1)} \qquad l = 1, 3 \tag{4.13}$$

The dispersion relation associated with the plate is now derived from equations (4.6) by applying traction free boundaries boundary conditions (2.7) and (2.8) at the upper and lower faces $x_3 = \pm d$ of the plate, thus

$$\sum_{l=1}^{2} r_{33(l)} (U^{(2l-1)} e^{i\xi\alpha_l d} + U^{(2l)} e^{-i\xi\alpha_l d}) = 0 \tag{4.15}$$

$$\sum_{l=1}^{2} r_{33(l)} (U^{(2l-1)} e^{-i\xi\alpha_l d} + U^{(2l)} e^{i\xi\alpha_l d}) = 0 \tag{4.16}$$

$$\sum_{l=1}^{3} r_{13(l)} (U^{(2l-1)} e^{i\xi\alpha_l d} - U^{(2l)} e^{-i\xi\alpha_l d}) = 0 \tag{4.17}$$

$$\sum_{l=1}^{2} r_{13(l)} (U^{(2l-1)} e^{-i\xi\alpha_l d} - U^{(2l)} e^{i\xi\alpha_l d}) = 0 \tag{4.18}$$

On further simplifying equations (4.15)- (4.18), we have

$$\sum_{l=1}^{2} r_{33(l)} (\tilde{U}_l^+ C_l + i\tilde{U}_l^- S_l) = 0 \tag{4.19}$$

$$\sum_{l=1}^{2} r_{33(l)} (\tilde{U}_l^+ C_l - i\tilde{U}_l^- S_l) = 0 \tag{4.20}$$

$$\sum_{l=1}^{2} r_{13(l)} (\tilde{U}_l^- C_l + i\tilde{U}_l^+ S_l) = 0 \tag{4.21}$$

$$\sum_{l=1}^{2} r_{13(l)} (\tilde{U}_l^- C_l - i\tilde{U}_l^+ S_l) = 0 \tag{4.22}$$

The symmetry of the plate allows us to simplify the system of four homogeneous equations in four unknowns into two systems of two equations in two unknowns, which on employing straight forward algebraic manipulations, yield the following relations associated with the plate

$$\sum_{l=1}^{2} r_{33(l)} \tilde{U}_l^+ C_l = 0 \tag{4.23}$$

$$\sum_{l=1}^{2} r_{13(l)} \tilde{U}_l^+ S_l = 0 \tag{4.24}$$

and

$$\sum_{l=1}^{2} r_{33(l)} \tilde{U}_l^- S_l = 0 \quad (4.25)$$

$$\sum_{l=1}^{2} r_{13(l)} \tilde{U}_l^- C_l = 0 \quad (4.26)$$

within which

$$C_l = \cos(\xi \alpha_l d), \; S_l = \sin(\xi \alpha_l d)$$

$$\tilde{U}_l^+ = U^{(2l-1)} + U^{(2l)}, \tilde{U}_l^- = U^{(2l-1)} - U^{(2l)} \quad (4.27)$$

The condition that the system of equations (4.23)-(4.24) and (4.25)-(4.26) admit a non-trivial solution give rise to the dispersion relations associated symmetric with and antisymmetric waves respectively.

5. ANTISYMMETRIC WAVES

The dispersion relation associated with flexural waves equation is obtained by taking $U^{(2l-1)} = U^{(2l)}$, thus $\bar{u}_1$, and $\bar{u}_3$ have the form

$$\bar{u}_1 = 2\sum_{l=1}^{2} U^{(2l)} C_l , \; \bar{u}_3 = 2i \sum_{l=1}^{2} q_{3(l)} U^{(2l)} S_l , \quad (5.1)$$

and therefore require that system of equation (4.25)- (4.26) admit a non-trivial solution provided the determinant of coefficients associated with these equations vanishes, which after a little and straight forward algebraic manipulation, may cast in the form

$$r_{33(1)} r_{13(2)} \Gamma_1 + r_{33(2)} r_{13(1)} \Gamma_2 = 0 \quad (5.2)$$

where

$$\Gamma_l = \tan(\gamma \alpha_l) \text{ and } \gamma = \xi d = \frac{\omega}{c} \quad (5.3)$$

6. SYMMETRIC WAVES

The dispersion relation associated with symmetric waves equation is obtained by taking $U^{(2l-1)} = -U^{(2l)}$, and determinant of coefficients of (4.23)- (4.24) yields the dispersion relation associated with extensional waves, namely

$$r_{33(1)} r_{13(2)} \Gamma_2 \Gamma_3 + r_{33(2)} r_{13(1)} \Gamma_1 \Gamma_3 = 0 \quad (6.1)$$

thus $\bar{u}_1$, and $\bar{u}_3$ have the form

$$\bar{u}_1 = -2i \sum_{l=1}^{2} U^{(2l)} S_l , \; \bar{u}_3 = -2 \sum_{l=1}^{2} q_{3(l)} U^{(2l)} C_l , \quad (6.2)$$

G_1, G_2 and Γ_l are defined in (5.3)

7. SPECIAL CASES

Cubic and Isotropic Materials:

Results for materials possessing transverse isotropy, cubic symmetry and isotropic case, can be easily obtained from equations (5.2) and (6.1) by imposing the additional conditions on the constants, namely

$$c_{33} = c_{22}, \; c_{13} = c_{12}, \; c_{55} = c_{66}, \; c_{22} - c_{23} = 2c_{44}, \quad (7.5)$$

and for cubic symmetry

$$c_{11} = c_{22} = c_{33}, \; c_{13} = c_{12} = c_{23}, c_{44} = c_{55} = c_{66} \quad \text{(cubic)} \quad (7.6)$$

Finally, for the isotropic case

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu, \; c_{13} = c_{12} = c_{23} = \lambda, \; c_{44} = c_{55} = c_{66} = \mu \quad \text{(isotropic)} \quad (7.7)$$

8. NUMERICAL RESULTS AND DISCUSSION

Numerical illustrations of the analytical characteristic equations are presented in the form of dispersion curves. These curves are obtained by keeping ξ (wave number) real and letting c be complex. Then the phase velocity is defined as Re(c), and the imaginary part of c is measure the damping of the waves. One can also let c be real and let ξ be complex. In this case the wave c corresponding to Re(ξ), and Im(ξ) is a measure of the attenuation of the wave. To find the solutions of a characteristic equation, Mathcad software is used to solve it as an analytic function by considering material, given with the following properties

$$c_1 = 0.3851,\ c_2 = 0.2365,\ c_3 = 0.5485,\ c_{11} = 1.628 \times 10^{11}\ Nm^{-2},\ \rho = 7.14 \times 10^{3}\ kmg^{-3},$$

In figures 1 and 2, dispersion curves corresponding to few lower modes of antisymmetric and symmetric wave modes in the forms of variations of phase velocity (dimensionless) with wave number (dimensionless) are plotted at $\theta = \pi/2$. It is obvious that the largest value corresponds to the quasi-longitudinal mode. Higher modes appear in both the cases (antisymmetric and symmetric) with ξ increases. One of these seems to be associated with rapid change in the slope of the mode. Lower modes of antisymmetric and symmetric are found more influenced at low values of wave number.

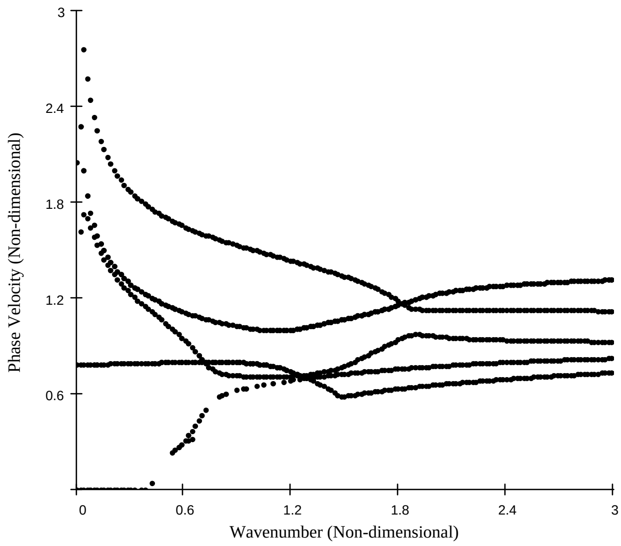

Fig.1. Phase velocity Vs wavenumber antisymmetric wave modes for transversely isotropic plate

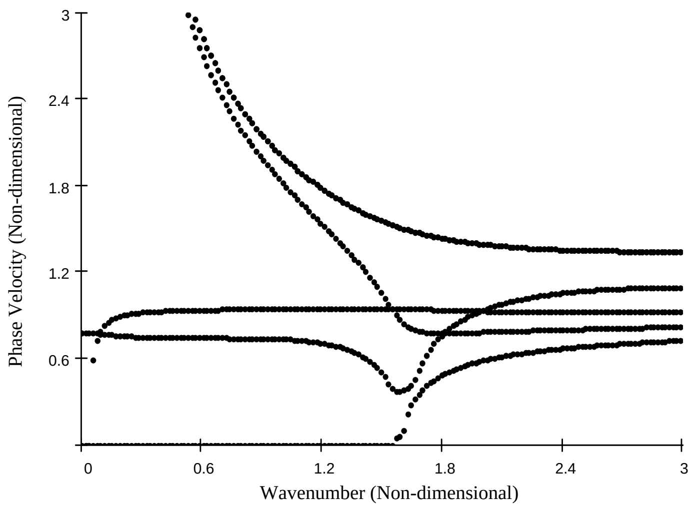

Fig.2. Phase velocity Vs wavenumber symmetric wave modes for transversely isotropic plate

Dispersion curves in figure 1 correspond to antisymmetric wave modes and figure 2, to symmetric wave modes. The phase velocity of the lowest antisymmetric and symmetric mode is observed to increase from zero value at zero wave number limits, and tends towards Rayleigh velocity asymptotically with an increase in wave number. The phase velocities of higher modes of propagation, antisymmetric and symmetric, attain quite large values at vanishing wave number.

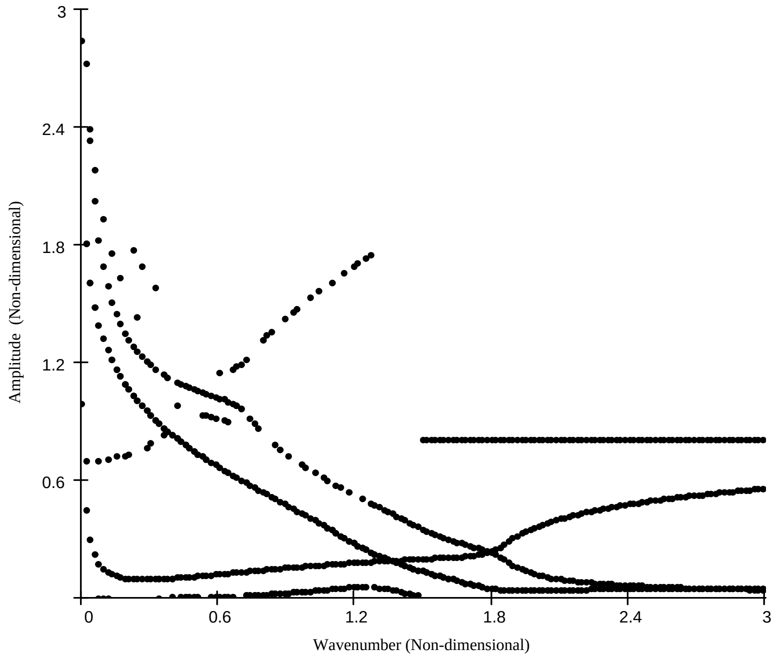

Fig.3. Amplitude Vs wavenumber antisymmetric wave modes for transversely isotropic plate

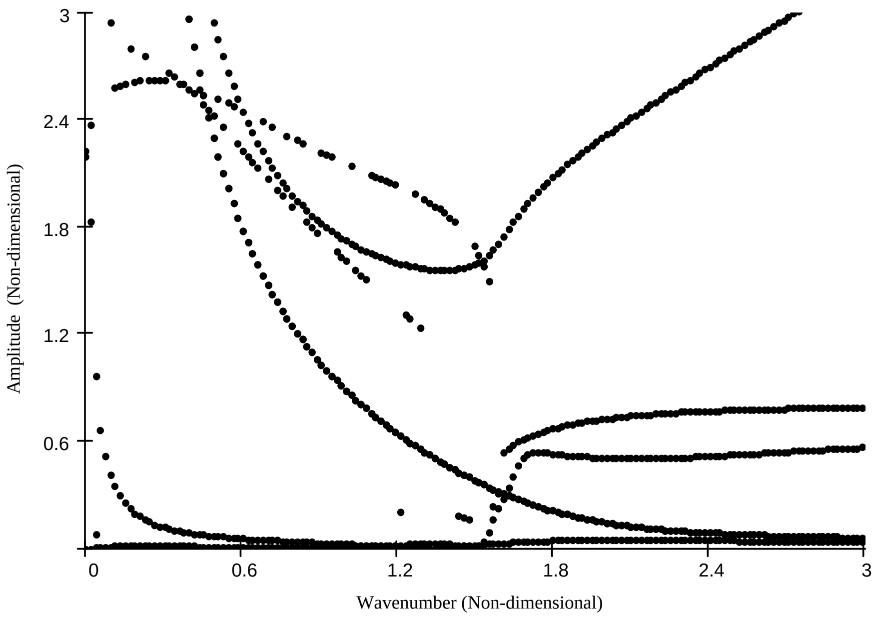

Fig.4. Amplitude Vs wavenumber symmetric waves for transversely isotropic plate

Lowest antisymmetric modes have non zero and the lowest symmetric modes have zero velocity at vanishing wave numbers, but the phase velocity of these modes also become asymptotically close to the surface wave velocity with increasing value of the wave number. The behavior of higher modes of propagation is observed to be similar to other cases. In figures 3 and 4 depict the variation attenuation of the few antisymmetric and symmetric lower wave modes. It is again observed that at zero wave number limits attenuation is high and it decrease with increasing value of the wave number.

9. CONCLUSIONS

Analysis for the propagation of plane harmonic waves in an infinite homogeneous orthotropic plate of finite thickness times is studied .The horizontally polarized SH wave (3.7) gets decoupled from the rest of motion and propagates without dispersion or damping on the same plate. Waves namely, quasi-longitudinal and quasi-transverse are found coupled with each other due to the anisotropic effects. The phase velocity of the waves gets modified due to the anisotropic effects. The dispersion characteristics for antisymmetric and symmetric wave's modes have been taken into consideration. Phase velocity of these modes also become asymptotically close to the surface wave velocity with increasing value of the wave number. It is also accomplished that attenuation is high at zero wave number limits and it decrease with increasing value of the wave number

ACKNOWLEDGMENTS
This work is supported by the Council of Scientific and Industrial Research, Extramural Research Division, CSIR Complex New Delhi INDIA, 110012 under research grant No. 22(0374)/04/EMR-II.

REFERENCES

[1.] *H. LAMB, On Waves in an Elastic Plate, Proceedings of the Royal Society, Vol. A93 (1917), 114-128.*
[2.] *I. A. VIKTOROV, Rayleigh and Lamb Waves , Plenum , New York, 1967*
[3.] *E. DIEULESAINT and D. ROYER, Elastic Waves in Solids, John Wiley and Sons, New York, 1980.*
[4.] *H. E. EKSTEIN, High Frequency Vibrations of Thin Crystal Plates, Phys. Review, 68(1945), 11-23.*
[5.] *Y. H. PAO and R. K. KAUL. Waves and Vibrations in Isotropic and Anisotropic Plates, Pergamon, New York, pp. 149-195, 1974.*
[6.] *L. P. SOLIE and B. A. AULD, Elastic Waves in Free Anisotropic Plates J. Acoust. Soc. Am. 54 (1973), 50-65.*
[7.] *J. L. SYNGE, Elastic Waves in anisotropic Media, J. Math. Phy., Vol. 25, 322-334, 1957.*
[8.] *G. R. LIU., J., TANI, K. WATANABE, and T., OHYOSHI, Lamb Wave Propagation in Anisotropic Laminates, ASME Journal of Applied Mechanics, Vol. 57(1990), 923-929.*
[9.] *S. K. DATTA, A. H. SHAH, R. I. BRATTON and T. CHAKRABOTY, Wave Propagation in laminated composite plates, J. Acoust. Soc. Am. Vol. 83(6) (1989). 2020-2026.*
[10.] *V. DAYAL and V. K. KINRA, Leaky Lamb waves in anisotropic plate I: An exact solution and experiments, J. Acoust. Soc. Am. Vol. 83(6) (1989), 2268-2276.*
[11.] *I. ABUBAKAR, Free Vibration of a transversely isotropic Plate, Quart. J. of Mech. and Appl. Mathematics, Vol. 15(1962), 129-136.*
[12.] *A. H. NAYFEH and D. E. CHIMENTI, Free Wave propagation in plates of general anisotropic media, J. Applied Mechanics, Vol.87 (1989), 881-886.*
[13.] *K. L. VERMA, Propagation of Generalized Free waves in plates of anisotropic media. Proc. Sec. Int. Symp. on thermal stresses and related topics 199-201, 1997.*
[14.] *A. H. NAYFEH and D. E. CHIMENTI, Free Wave propagation in plates of general anisotropic media, J. applied Mechanics, Vol.87 (1989), 881-886.*
[15.] *Y. LI and R. B. THOMPSON, Influence of anisotropy on the dispersion of guided ultrasonic plate modes, J. Acoust. Soc. Am. Vol. 87(1990), 1911-1931.*