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STUDY ON THE CONSTRUCTION OF COMPLEX 3D SHAPES WITH SUPERELLIPSOIDS AND SUPERTOROIDS

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ABSTRACT: This paper presents a CAD study for generating of complex shapes with superellipsoids and supertoroids based on computational geometry. The Madsie Freestyle 1.5.3 application for computation helps in obtaining conclusions referring to shape and profile for different 3D complex objects, but also facilitate the design of new 3D models. Results from this study are applied in geometric constructions and computer aided design used in engineering and sculpture design.
KEYWORDS: engineering design, sculpture design, superellipsoid, supertoroid, implicit surface

INTRODUCTION

Research in engineering and sculpture design involve considerable mathematical modeling to solve a wide variety of problems [1, 2, 3]. A great diversity of geometry representations has been used for reconstruction, modeling, editing, and rendering of 3D objects [4, 5, 6, 7, 8].

The sculptured surfaces, or so-called free-form surfaces, for aesthetic and/or functional reasons, have usually free-formed geometry of complex shapes and are difficult to be machined.

SUPERELLIPSOID AND SUPERTOROID

In 1981, Barr generalized the superellipsoids to a family of 3D shapes that he named superquadrics and also introduced the notation common in the computer vision literature [9]. He demonstrated that superquadric models, in particular for CAD design, have compactly represented continuum useful forms with rounded edges that can easily be rendered and shaded and further deformed by parametric deformations [10].

Superquadrics constitute a class of surfaces which possess a natural parametric and implicit description. The superquadrics are: the superellipsoid, the superhyperboloid of one and two sheets, and the supertoroid [11]. In last decade, superellipsoids have received significant attention for object modeling in computer vision and computer graphics [10, 11, 12].

a) SUPERELLIPSOID. A superellipsoid, as an ellipsoid's extension, is the result of the spherical product of two 2D models (two superellipses) [12]. A superellipse, analogous to a circle, is expressed as [10]:

$$\left(\frac{x}{a}\right)^{2/\varepsilon}+\left(\frac{y}{b}\right)^{2/\varepsilon}=1,\ a>0, b>0. \tag{1}$$

where exponentiation with ε is a signed power function such that:

$$\cos^{\varepsilon}\theta = sign(\cos\theta)\left|\cos\theta\right|^{\varepsilon}. \tag{2}$$

Superellipsoids can be expressed by a spherical product of a pair of such superellipses [10]:

$$r(\eta,\omega)=s_1(\eta)\otimes s_2(\omega)=\begin{bmatrix}\cos^{\varepsilon 1}\eta\\ a_3\sin^{\varepsilon 1}\eta\end{bmatrix}\otimes\begin{bmatrix}a_1\cos^{\varepsilon 2}\omega\\ a_2\sin^{\varepsilon 2}\omega\end{bmatrix}=\begin{bmatrix}a_1\cos^{\varepsilon 1}\eta\cos^{\varepsilon 2}\omega\\ a_2\cos^{\varepsilon 1}\eta\sin^{\varepsilon 2}\omega\\ a_3\sin^{\varepsilon 1}\eta\end{bmatrix},\ -\frac{\pi}{2}\le\eta\le\frac{\pi}{2};\ -\pi\le\omega\le\pi. \tag{3}$$

The a_1, a_2, a_3 parameters are scaling factors along the three coordinate axes. ε_1 and ε_2 are derived from the exponents of the two original superellipses.

This flexibility achieved by raising each trigonometric term to an exponent is of particular interest to us. In simple terms, these exponents, control the relative roundness and squareness in both the horizontal and vertical directions.

The shape of the superellipsoid cross section parallel to the [*xoy*] plane is determined by ε_2, while the shape of the superellipsoid cross section in a plane perpendicular to the [*xoy*] plane and containing *z* axis is determined by ε_1.

A superellipsoid is defined as the solution of the general form of the implicit equation [11]:

$$\left(\left(\frac{x}{a_1}\right)^{2/\varepsilon_2}+\left(\frac{y}{a_2}\right)^{2/\varepsilon_2}\right)^{\varepsilon_2/\varepsilon_1}+\left(\frac{z}{a_3}\right)^{2/\varepsilon_1}=1. \qquad (4)$$

All points with coordinates (*x*, *y*, *z*) that correspond to the above equation lie on the surface of the superellipsoid. This is a compact model defined by only five parameters that permits to handle different shapes.

The exponent functions are continuous to ensure that the superellipsoid model deforms continuously and thus has a smooth surface. This form provides information on the position of a 3D point related to the superellipsoid surface, which is important for interior/exterior determination [9, 12]. We have an inside-outside function F (x, y, z):

- *F* (*x*, *y*, *z*) = 1, when the point lies on the surface;
- *F* (*x*, *y*, *z*) < 1, when the point is inside the superellipsoid;
- *F* (*x*, *y*, *z*) > 1, when the point is outside.

The existence of the inside-outside functions means that superquadrics can be manipulated by means of solid boolean operations, such as union, intersection, and subtraction [9].

b) SUPERTOROID. A supertoroid can be defined by the surface vector [10]:

$$r(\eta,\omega)=s_3(\eta)\otimes s_4(\omega)=\begin{bmatrix}a_4+\cos^{\varepsilon 1}\eta\\ a_3\sin^{\varepsilon 1}\eta\end{bmatrix}\otimes\begin{bmatrix}a_1\cos^{\varepsilon 2}\omega\\ a_2\sin^{\varepsilon 2}\omega\end{bmatrix}=\begin{bmatrix}a_1(a_4+\cos^{\varepsilon 1}\eta)\cos^{\varepsilon 2}\omega\\ a_2(a_4+\cos^{\varepsilon 1}\eta)\sin^{\varepsilon 2}\omega\\ a_3\sin^{\varepsilon 1}\eta\end{bmatrix},\quad \begin{cases}-\pi\le\eta\le\pi\\ -\pi\le\omega\le\pi\end{cases} \qquad (5)$$

The a_1, a_2, a_3 parameters are scaling factors along the three coordinate axes. ε_1 and ε_2 are derived from the exponents of the a superellipse and another superellipse with $center_x > a_g$.

The shape of the supertoroid cross section parallel to the [*xoy*] plane is determined by ε_2, while the shape of the supertoroid cross section in a plane perpendicular to the [*xoy*] plane and containing *z* axis is determined by ε_1. A supertoroid is the spherical product of a superellipse and another superellipse with $center_x > a_g$ and is defined by the implicit equation [11]:

$$\left(\left(\left(\frac{x}{a_1}\right)^{2/\varepsilon_2}+\left(\frac{y}{a_2}\right)^{2/\varepsilon_2}\right)^{\varepsilon_2/\varepsilon_1}-a_4\right)^{2/\varepsilon_1}+\left(\frac{z}{a_3}\right)^{2/\varepsilon_1}-1=0. \qquad (6)$$

where a_4 is a positive real offset value which is related to the radius of the supertoroid *R* in the following way:

$$a_4=R\cdot\left(a_1^{2}+a_2^{2}\right)^{-1/2}. \qquad (7)$$

All points with coordinates (*x*, *y*, *z*) that correspond to the above equation lie on the surface of the supertoroid. This is a compact model defined by only five parameters that permits to handle a large variety of shapes. This form provides information on the position of a 3D point related to the supertoroid surface, which is important for interior/exterior determination [9]. We have an inside-outside function *F* (*x*, *y*, *z*):

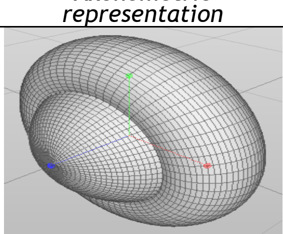
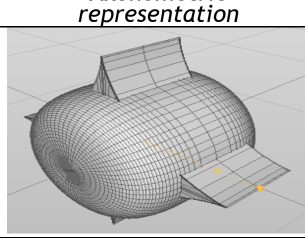
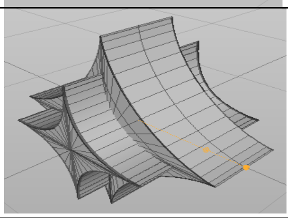
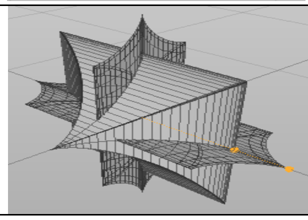
- *F* (*x*, *y*, *z*) = 1 when the point lies on the surface;
- *F* (*x*, *y*, *z*) < 1 when the point is inside the supertoroid;
- *F* (*x*, *y*, *z*) > 1 when the point is outside.

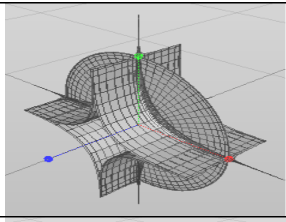
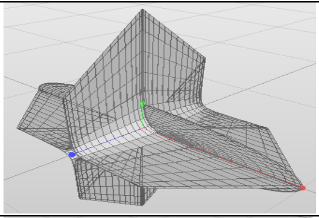
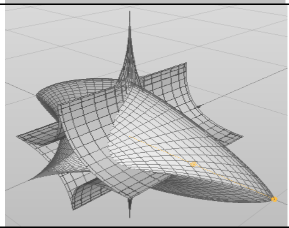
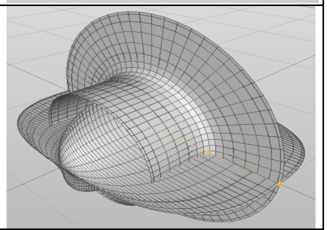
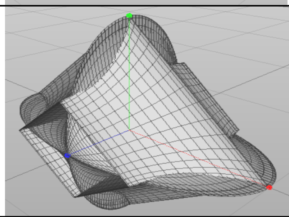
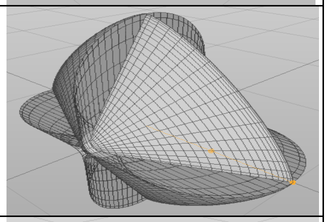
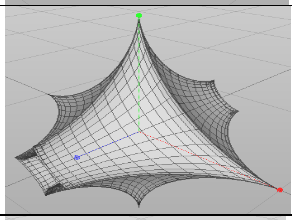
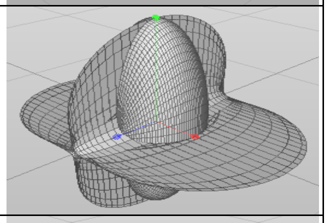
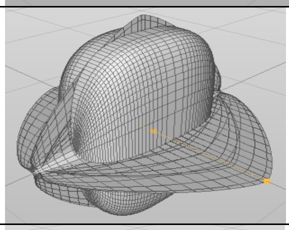
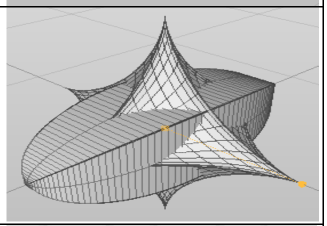
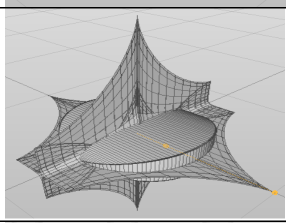
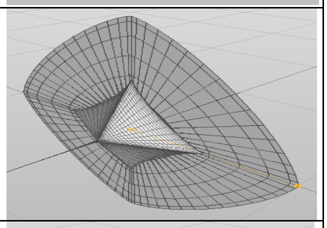
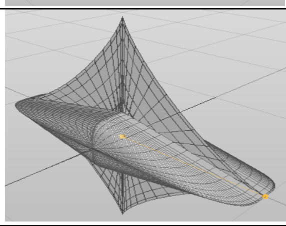
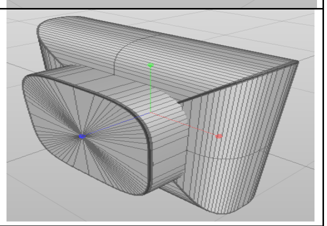
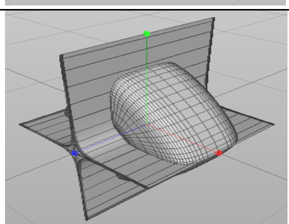
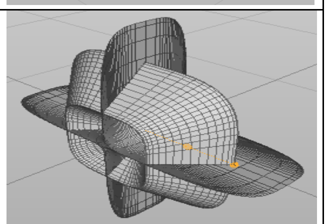
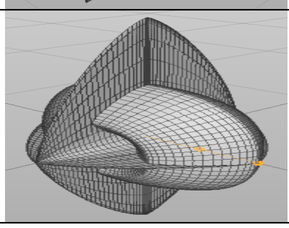
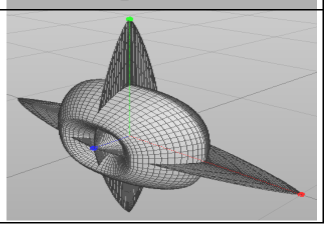
Superquadrics are an easy class of objects to use because they have well defined normal and tangent vectors. Normal vectors are used in intensity calculations during rendering. Both the normal and tangent vectors are used to calculate the curvature of the surface [13].

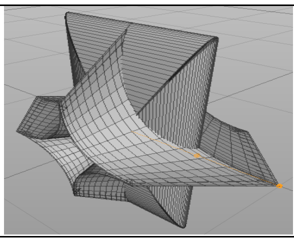
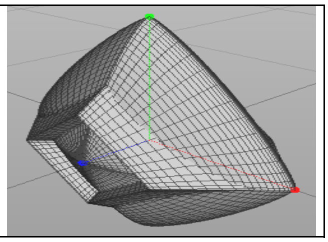
GRAPHICAL REPRESENTATIONS

The Madsie Freestyle 1.5.3 application for computation was used to generate the 3D objects with superellipsoids and supertoroids [14] and the graphical representations are given in Table 1.

Table 1. Graphical representations of 3D complex objects

No.	*Values of parameters*	*Axonometric representation*	No.	*Values of parameters*	*Axonometric representation*
1	*Superellipsoid:* $r_1 = 1$, $r_2 = 1$ $r_3 = 1$ $e_1 = 1$, $e_2 = 1$ *Supertoroid:* $r_4 = 1$, $r_5 = 0.5$ $e_3 = 1$, $e_4 = 1$		*2*	*Superellipsoid:* $r_1 = 1$, $r_2 = 1$ $r_3 = 1$ $e_1 = 1$, $e_2 = 0.5$ *Supertoroid:* $r_4 = 1$, $r_5 = 0.5$ $e_3 = 3$, $e_4 = 0.01$	
3	*Superellipsoid:* $r_1 = 1$, $r_2 = 1$ $r_3 = 1$ $e_1 = 4$, $e_2 = 0.01$ *Supertoroid:* $r_4 = 1$, $r_5 = 0.5$ $e_3 = 3$, $e_4 = 0.01$		*4*	*Superellipsoid:* $r_1 = 1$, $r_2 = 1$ $r_3 = 1$ $e_1 = 0.01$, $e_2 = 3$ *Supertoroid:* $r_4 = 1$, $r_5 = 0.5$ $e_3 = 8$, $e_4 = 3$	

5	Superellipsoid: $r_1 = 1$, $r_2 = 1$ $r_3 = 1$ $e_1 = 1$, $e_2 = 8$ Supertoroid: $r_4 = 1$, $r_5 = 0.5$ $e_3 = 5$, $e_4 = 8$		6	Superellipsoid: $r_1 = 2$, $r_2 = 0.5$ $r_3 =$ 1, $e_1 = 1$, $e_2 = 8$ Supertoroid: $r_4 = 1$, $r_5 = 1$ $e_3 = 8$, $e_4 = 2$	
7	Superellipsoid: $r_1 = 2$, $r_2 = 1$ $r_3 = 1.5$ $e_1 = 1.5$, $e_2 = 3$ Supertoroid: $r_4 = 1$, $r_5 = 1$ $e_3 = 4$, $e_4 = 8$		8	Superellipsoid: $r_1 = 2$, $r_2 = 1$ $r_3 = 1.5$, $e_1 = 4.5$, $e_2 = 0.7$ Supertoroid: $r_4 = 1$, $r_5 = 1$ $e_3 = 1$, $e_4 = 8$	
9	Superellipsoid: $r_1 = 2$, $r_2 = 2$ $r_3 = 1$ $e_1 = 4$, $e_2 = 1$ Supertoroid: $r_4 = 1$, $r_5 = 1$ $e_3 = 2$, $e_4 = 3$		10	Superellipsoid: $r_1 = 2$, $r_2 = 2$ $r_3 = 1$ $e_1 = 1.5$, $e_2 = 2$ Supertoroid: $r_4 = 1$, $r_5 = 1$ $e_3 = 8$, $e_4 = 1$	
11	Superellipsoid: $r_1 = 2$, $r_2 = 2$ $r_3 = 1$ $e_1 = 8$, $e_2 = 8$ Supertoroid: $r_4 = 0.5$, $r_5 = 1.5$ $e_3 = 3$, $e_4 = 3$		12	Superellipsoid: $r_1 = 0.5$, $r_2 = 1.5$ $r_3 = 0.5$, $e_1 = 1$, $e_2 = 1$ Supertoroid: $r_4 = 0.5$, $r_5 = 1$ $e_3 = 8$, $e_4 = 1$	
13	Superellipsoid: $r_1 = 0.5$, $r_2 = 1.5$ $r_3 = 1$, $e_1 = 0.5$, $e_2 = 0.7$ Supertoroid: $r_4 = 0.2$, $r_5 = 1.5$ $e_3 = 8$, $e_4 = 1.5$		14	Superellipsoid: $r_1 = 0.5$, $r_2 = 0.5$, $r_3 = 1.5$ $e_1 = 0.01$, $e_2 = 1$ Supertoroid: $r_4 = 0$, $r_5 = 1.5$ $e_3 = 3$, $e_4 = 3$	
15	Superellipsoid: $r_1 = 1$, $r_2 = 0.25$ r_3 $= 1$, $e_1 = 0.01$, $e_2 = 1$ Supertoroid: $r_4 = 0.5$, $r_5 = 1.5$ $e_3 = 8$, $e_4 = 3$		16	Superellipsoid: $r_1 = 1$, $r_2 = 1$ $r_3 = 0.5$ $e_1 = 2.5$, $e_2 = 2$ Supertoroid: $r_4 = 0$, $r_5 = 2$ $e_3 = 1.5$, $e_4 = 8$	
17	Superellipsoid: $r_1 = 2$, $r_2 = 0.5$ $r_3 =$ 0.5, $e_1 = 2$, $e_2 = 0.5$ Supertoroid: $r_4 = 0$, $r_5 = 2$ $e_3 = 2.5$, $e_4 = 8$		18	Superellipsoid: $r_1 = 1$, $r_2 = 1$ $r_3 = 1$, $e_1 = 0.5$, $e_2 = 0.01$ Supertoroid: $r_4 = 1$, $r_5 = 0.5$ $e_3 = 0.01$, $e_4 = 0.7$	
19	Superellipsoid: $r_1 = 1$, $r_2 = 1.5$ $r_3 = 1$, $e_1 = 8$, $e_2 = 0.01$ Supertoroid: $r_4 = 0.5$, $r_5 = 0.5$ $e_3 = 1.5$, $e_4 = 0.5$		20	Superellipsoid: $r_1 = 1.5$, $r_2 = 1.5$ $r_3 = 0.5$, $e_1 = 8$, $e_2 = 0.5$ Supertoroid: $r_4 = 0.5$, $r_5 = 0.5$ $e_3 = 0.5$, $e_4 = 2$	
21	Superellipsoid: $r_1 = 1$, $r_2 = 2$ $r_3 = 1$, $e_1 = 8$, $e_2 = 1.5$ Supertoroid: $r_4 = 0.5$, $r_5 = 0.5$ $e_3 = 0.8$, $e_4 = 1.5$		22	Superellipsoid: $r_1 = 2$, $r_2 = 2$ $r_3 = 0.5$, $e_1 = 8$, $e_2 = 1.5$ Supertoroid: $r_4 = 0.5$, $r_5 = 0.5$, $e_3 = 0.8$, $e_4 = 1.5$	

23	*Superellipsoid:* $r_1 = 1$, $r_2 = 2$, $r_3 = 0.25$ $e_1 = 0.01$, $e_2 = 1$ *Supertoroid:* $r_4 = 1$, $r_5 = 1$ $e_3 = 4$, $e_4 = 2$		24	*Superellipsoid:* $r_1 = 1$, $r_2 = 1.2$, $r_3 = 0.5$ $e_1 = 1.7$, $e_2 = 1.5$ *Supertoroid:* $r_4 = 0.5$, $r_5 = 0.5$ $e_3 = 2$, $e_4 = 2$	

❖ CONCLUSIONS

This paper presents a CAD method for generation of complex shapes with superellipsoids and supertoroids.

The Madsie Freestyle 1.5.3 application for computation helps in obtaining conclusions referring to shape of complex 3D objects. Results from this study are applied in geometric constructions and computer aided design used in engineering and sculpture design.

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