ANNALS OF FACULTY ENGINEERING HUNEDOARA
- INTERNATIONAL JOURNAL OF ENGINEERING
Tome IX (Year 2011). Fascicule 3. (ISSN 1584 - 2673)

[1.] Ludovic Dan LEMLE, [2.]Tudor BÎNZAR, [3.]Lucian Flavius PATER, [4.]Valentina BALASOIU

RIESZ-DUNFORD REPRESENTATION THEOREM FOR UNIFORMLY CONTINUOUS SEMIGROUPS

[1.] UNIVERSITY "POLITEHNICA" TIMIŞOARA, FACULTY OF ENGINEERING HUNEDOARA, ROMÂNIA
[2-3.] UNIVERSITY "POLITEHNICA" TIMIŞOARA, DEPARTMENT OF MATHEMATICS, ROMÂNIA
[4.] NATIONAL COLLEGE OF INFORMATICS "TRAIAN LALESCU" HUNDEOARA, ROMANIA

ABSTRACT: This note presents a Riesz-Dunford type representation and a Bromwich type representation for uniformly continuous semigroups on a Banach space.
A.M.S.(2010) Subject Classification: 47D03.
KEYWORDS: uniformly continuous semigroups, Riesz-Dunford representation, Bromwich representation

❖ INTRODUCTION

Let *E* be a complex Banach space. We denote by *B(E)* the Banach algebra of bounded linear operators on *E*. For a closed linear operator *A*, not necessarily bounded, with domain *D(A)* in the space *E*, denote by $\rho(A)$ and $R(.\,,A)$ the resolvent set of *A* and the resolvent of *A*, respectively.

The family of operators $\{T(t)\}_{t\geq 0}\subset B(E)$ is said to be a *semigroup of bounded linear operators on E* if

(i) T(0)=I (I is the identity operator on *E*);
(ii) T(t+s)=T(t)T(s) for all t,s≥0 (the semigroup property).

The semigroup $\{T(t)\}_{t\geq 0}\subset B(E)$ is said to be *uniformly continuous* if $t\mapsto T(t)$ is continuous on $[0,\infty)$ in the uniform operator topology. Due to semigroup property this is equivalent to

$$\lim_{t\downarrow 0}\|T(t)-I\|=0 \quad .$$

The most important object associated to a C_0- semigroup $\{T(t)\}_{t\geq 0}$ is its infinitesimal generator. The linear operator *A* defined by

$$D(A)=\left\{x\in E:\lim_{t\downarrow 0}\frac{T(t)x-x}{t}\quad \text{exists}\right\}$$

and

$$Ax=\lim_{t\downarrow 0}\frac{T(t)x-x}{t}\quad ,\quad \forall x\in D(A)$$

is the *infinitesimal generator* of the semigroup $\{T(t)\}_{t\geq 0}$. Clearly the operator $A:D(A)\subseteq E\to E$ is linear but not necessarily bounded unsless *D(A)* is all of *E*. Nathan [4] and Yosida [7] proved that the infinitesimal generator of a semigroup is a bounded linear operator in *E* if and only if the semigroup is uniformly continuous. For more information about C_0- semigroup we refer to Davies [1], Hille and Phillips [2], Pazy [5], Yosida [8] and the references therein.

This paper is dedicated to the problem of representing the semigroup $\{T(t)\}_{t\geq 0}$ in terms of its infinitesimal generator. We can obtain the semigroup from the resolvent of the generator *A* by inverting the Laplace transform. Similar results for C_0-semigroups were presented in Lemle and Jiang [3].

❖ RIESZ-DUNFORD'S TYPE REPRESENTATIONS

In this section we give a Riesz-Dunford's type representation for uniformly continuous semigroups. For this purpose we use a special class of Jordan's curves for a bounded linear operator defined by Reghiş and Babescu [6].

2.1. Definition. *A Jordan closed smooth curve which surround* $\sigma(A)$ *is said to be A-spectral if it is homotope to a circle* C_r *of radius* $r>\|A\|$ *centered at the origin.*

We have:

2.2. Theorem. *Let A be the infinitesimal generator of the uniformly continuous semigroup* $\{T(t)\}_{t\geq 0}$. *If* Γ_A *is an A-spectral curve then*

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_A} e^{\lambda t} R(\lambda; A)d\lambda \quad , \quad \forall t \ge 0.$$

Proof. Let Γ_A be an A-spectral curve. Then Γ_A is homotope to the circle C_r of radius $r > \|A\|$ centered at the origin. We have:

$$\frac{1}{2\pi i}\int_{\Gamma_A} e^{\lambda t} R(\lambda; A)d\lambda = \frac{1}{2\pi i}\int_{C_r} e^{\lambda t} R(\lambda; A)d\lambda \quad , \quad \forall t \ge 0.$$

But the serie

$$R(\lambda; A) = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$$

converges uniformly for λ on compacts set of $\{\lambda \in C : |\lambda| > \|A\|\}$, particularly on circle C_r. Then

$$\frac{1}{2\pi i}\int_{C_r} e^{\lambda t} R(\lambda; A)d\lambda = \frac{1}{2\pi i}\int_{C_r} e^{\lambda t}\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}} d\lambda = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{C_r} \frac{e^{\lambda t}}{\lambda^{n+1}} d\lambda A^n.$$

Using the identities

$$\frac{1}{2\pi i}\int_{C_r} \frac{e^{\lambda t}}{\lambda^{n+1}} d\lambda = \frac{t^n}{n!} \quad , \quad \forall n \in N,$$

we conclude that

$$\frac{1}{2\pi i}\int_{\Gamma_A} e^{\lambda t} R(\lambda; A)d\lambda = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = T(t) \quad , \quad \forall t \ge 0.$$

❖ BROMWICH'S TYPE REPRESENTATION

Next theorem gives Bromwich's type representation theorem for uniformly continuous semigroups.

3.1. Theorem. *Let A be the infinitesimal generator of a* C_0*-semigroup* $\{T(t)\}_{t\ge 0}$. *If* $a > \|A\|$, *then*

$$T(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt} R(z; A)dz$$

and the integral converges uniformly for t in bounded intervals.

Proof. Let $a > \|A\|$. For $R > 2a$ we consider Jordan's closed smooth curve

$$\Gamma_R = \Gamma'_R \cup \Gamma''_R$$

where

$$\Gamma'_R = \{a + i\tau : \tau \in [-R, R]\}$$

and

$$\Gamma''_R = \left\{a + R(\cos\varphi + i\sin\varphi) : \varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$$

For $z \in \Gamma'_R$ we have

$$|z| = |a + i\tau| > a > \|A\|$$

and for $z \in \Gamma''_R$ we find $|z| = |a + R(\cos\varphi + i\sin\varphi)| = |a - [-R(\cos\varphi + i\sin\varphi)]| \ge |a - R| > \|A\|$. Therefore from $z \in \Gamma_R$ it follows $z \in \rho(A)$. Moreover, Γ_R is homotop to the circle C of radius *R-a* centered at the origin. Then Γ_R is an A-spectral curve and from theorem 2.2 it follows that

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma_R} e^{zt} R(z; A)dz \quad , \quad \forall t \ge 0$$

for every $R > 2a$. If we denote $I'_t(R) = \frac{1}{2\pi i}\int_{\Gamma'_R} e^{zt} R(z; A)dz$ *and* $I''_t(R) = \frac{1}{2\pi i}\int_{\Gamma''_R} e^{zt} R(z; A)dz$

we can see that $T(t) = I'(R) + I''(R) \quad , \quad \forall t \ge 0.$

Next we show that

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\Gamma''_R} e^{zt} R(z; A)dz = 0$$

uniformly for *t* in bounded intervals. To this end we use the serie

$$R(z;A)=\sum_{n=0}^{\infty}\frac{A^n}{z^{n+1}}$$

which converges uniformly for *z* on compacts set of $\{z\in C:|z|>\|A\|\}$, particularly on Γ''_R. For every $R>2a$ and every $t\ge 0$ we have

$$I''(R)=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z^{n+1}}A^n dz\right)=\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z}dz\right)I+\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z^{n+1}}dz\right)A^n .$$

We consider $A_t(R)=\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z}dz\right)I$ and $B_t(R)=\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z^{n+1}}dz\right)A^n$

Changing variables to

$$z=a+R(\cos\varphi+i\sin\varphi)\quad,\quad \varphi\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$$

we obtain

$$A_t(R)=\left[\frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{e^{t(a+R\cos\varphi+i\sin\varphi)}}{z}R(-\sin\varphi+i\cos\varphi)d\varphi\right]I=\left[\frac{R}{2\pi}e^{ta}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{tR\cos\varphi}e^{itR\sin\varphi}\frac{1}{z}(\cos\varphi+i\sin\varphi)d\varphi\right]I$$

from where one deduce that

$$\|A_t(R)\|\le\frac{R}{2\pi}e^{ta}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left|e^{tR\cos\varphi}\right|\left|e^{itR\sin\varphi}\right|\frac{1}{|z|}|\cos\varphi+i\sin\varphi|d\varphi\le$$

$$\le\frac{R}{2\pi}e^{ta}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{tR\cos\varphi}\frac{1}{R-a}d\varphi\le\frac{1}{2\pi}\frac{R}{R-a}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{tR\cos\varphi}d\varphi$$

since for $z\in\Gamma''_R$ we have $|z|=|a+R(\cos\varphi+i\sin\varphi|>R-a$

therefore $\frac{1}{|z|}<\frac{R}{R-a}$.

Consider $0<t_1<t_2$ and $t\in[t_1,t_2]$. From the inequality $R>2a$, it follows that $2R-2a>R$ and therefore $\frac{R}{R-a}<2$.

Consequently $\|A_t(R)\|\le\frac{1}{\pi}e^{ta}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{tR\cos\varphi}d\varphi\le\frac{1}{\pi}e^{t_2 a}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{t_1R\cos\varphi}d\varphi$

But for every $\varphi\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ one obtain $e^{t_1R\cos\varphi}\le 1$ and we have

$$\lim_{R\to\infty}e^{t_1R\cos\varphi}=0 .$$

By Lebesgue's bounded convergences theorem it follows

$$\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}e^{t_1R\cos\varphi}d\varphi=0$$

so we deduce that $\lim_{R\to\infty}A_t(R)=0$ and the limit is uniform for $t\in[t_1,t_2]$.

We consider now the integral

$$B_t(R)=\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{\Gamma''_R}\frac{e^{zt}}{z^{n+1}}dz\right)A^n$$

For every $t\in[t_1,t_2]$ and $R>2a$ we have $e^{tR\cos\varphi}\le 1\quad,\quad\forall\varphi\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$, so that $\left|\int_{\Gamma''_R} \frac{e^{zt}}{z^{n+1}} dz\right| \le \frac{Re^{ta}}{(R-a)^{n+1}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\varphi} d\varphi \le \pi e^{ta} \frac{R}{(R-a)^{n+1}}$

Since $R > 2a > a + \|A\|$, it follows

$$\|B_t(R)\| \le \sum_{n=1}^{\infty} \frac{\|A\|^n}{2\pi} \left|\int_{\Gamma''_R} \frac{e^{zt}}{z^{n+1}} dz\right| \le \frac{e^{ta}}{2} \frac{R}{R-a} \sum_{n=1}^{\infty} \left(\frac{\|A\|}{R-a}\right)^n$$

and because $\frac{\|A\|}{R-a} < 1$

one deduce that $\|B_t(R)\| \le e^{ta} \frac{\|A\|}{2} \frac{R}{R-a} \frac{1}{R-a-\|A\|}$.

Consequently $\lim_{R\to\infty} B_t(R) = 0$ and the limit is uniform for $t \in [t_1, t_2]$.

Then we have $\lim_{R\to\infty} I''(R) = 0$ uniformly for $t \in [t_1, t_2]$ from where we conclude that

$$T(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma'_R} e^{zt} R(z;A) dz = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{zt} R(z;A) dz$$

and the integral converges uniformly for $t \in [t_1, t_2]$.

❖ REFERENCES

[1.] Davies, E.B. (1980). *One-parameter semigroups.* Academic Press.

[2.] Hille, E. and Phillips, R.S. (1957). *Functional Analysis and Semi-Groups.* A.M.S. Providence.

[3.] Lemle, L.D., Jiang, Y. (2007). Bromwich's type representation for semigroups of linear operators, *The Cyprus Journal of Sciences,* Vol. 5, 107-125.

[4.] Nathan, D.S. (1935). One parameter semigroups of transformations in abstract vector spaces, *Duke Math. J.,* 518-526.

[5.] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer Verlag.

[6.] Reghiş, M. and Babescu, Gh. (1980). Cosine and sine functions valued in Banach algebras, *Analele Univ. Timişoara,* Vol. 1, 83-92.

[7.] Yosida, K. (1936). On the group embedded in the metrical complete ring, *Japan J. Math.,* 7-26.

[8.] Yosida, K. (1967). *Functional Analysis.* Springer Verlag

ANNALS OF FACULTY ENGINEERING HUNEDOARA
– INTERNATIONAL JOURNAL OF ENGINEERING

Faculty of Engineering Hunedoara,
5, Revolutiei, 331128, Hunedoara,
ROMANIA
http://annals.fih.upt.ro