ANNALS OF FACULTY ENGINEERING HUNEDOARA
– INTERNATIONAL JOURNAL OF ENGINEERING
Tome IX (Year 2011). Extra Fascicule. (ISSN 1584 – 2673)

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THE LAGRANGE INTERPOLATION FORMULA FOR ANALYZING FLUID MOVEMENT IN NETWORK PROFILES

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ABSTRACT: The paper presents two calculus algorithms for the study of the compressible fluid's stationary movement through profile grids on an axial–symmetric flow–surface. The first method is based on an iterative formula developed by the authors to calculate the complex conjugate velocity (using the CVBEM algorithm). The second method solves the fundamental integral equation in real values by a priori building up the velocity potential's integral equation (BEM method). In this case it is presented the necessity of using the Lagrangian interpolation formula through five points for the calculation of the derivatives of the velocity potential. In both cases the consecutive approximations can be organized simultaneously or successive with respect to parameters ς (fluid's density) and h (thickness variation of fluid stratum).
KEYWORDS: hydrodynamic networks, boundary element method, Lagrange interpolation, complex velocity, velocity potential, Fredholme integral equation

❖ INTRODUCTION

We study the direct problem from the hydrodynamics of networks for the stationary subsonic movement of the compressible fluid through profile grids on an axial-symmetric flow-surface, in variable thickness of stratum.

The movement is completely defined by the following equations [6]:

❖ Continuity equation for the compressible fluid:

$$div(\varsigma \cdot h \cdot \bar{v}) = 0 \tag{1}$$

❖ Potential (irrational) movement equation:

$$rot\, \bar{v} = 0 \tag{2}$$

❖ Characteristic equation of state of compressible fluid:

$$\varsigma = \varsigma(p) \tag{3}$$

where: - ς and p are the fluid's density and pressure, respectively;

- $\bar{v}$ is the fluid's absolute velocity;
- *h* is a function of the thickness variation of fluid stratum.

Additionally to equations (1), (2), (3), while studying the direct problem, the following boundary value conditions are also considered:

a. The upstream and downstream velocities ($\vec{V}_{1\infty}$ and $\vec{V}_{2\infty}$, respectively) of the fluid are considered to be known;
b. The relative velocity on a flow-surface, thus also on the base profile L_0, is a tangential velocity. The transport velocity $\vec{u}$ is tangent to the circle that contains the current point of L_0. Hence, the considered flow-surface is a flow-surface for the absolute motion, thus we have:

$$(\vec{v} \cdot \vec{n})\big|_{L_0} = (\vec{w} + \vec{u}) \cdot \vec{n}\big|_{L_0} = (\vec{w} \cdot \vec{n})\big|_{L_0} = 0 \tag{4}$$

where: - $\vec{n}$ is the normal versor to the flow-surface;
- $\vec{w}$ is the fluid's relative velocity.

c. For ensuring unique solution, it is assumed that in the motion without detachments an equivalent condition with the Jukovski-Ciaplighin hypothesis is fulfilled, e.g. the equality of velocities in two points A' and A'' symmetrically situated on the trailing edge, thus we have [6]:

$$\left|\vec{v}_{A'}\right| = \left|\vec{v}_{A''}\right| \tag{5}$$

❖ PREZENTING THE PROBLEM. THE COMPLEX VELOSITY OF MOVEMENT

The fundamental equations from the CVBEM method [3] in the problem of the compressible fluid's movement on a axial-symmetric flow-surface, in variable thickness of stratum are [6, 7, 8]:

$$\begin{aligned} \overline{w}(z) &= \overline{V}_m + \int_{L_0} H(z,\xi)\overline{w}(\xi)d\xi + i\iint_{D^-_{0^*}} H(z,\xi)\hat{q}(\xi)d\xi d\eta \\ F(z) &= \overline{V}_m \cdot z + \Gamma \cdot G(z,\xi_A) + \int_{L_0} H(z,\xi)F(\xi)d\xi + i\iint_{D^-_{0^*}} G(z,\xi)\hat{q}(\xi)d\xi d\eta \end{aligned} \tag{6}$$

where: $\overline{w}(z) = v_x - iv_y$ - is the complex conjugate velocity of motion; $F(z) = \varphi + i\psi$ is the complex potential of motion, where φ is the velocity potential and ψ is the flow rate function; *A* - is a fixed point on the base profile L_0; *t* - is the grid step; *Γ* - is the circulation around L_0; $\overline{V}_m = \frac{1}{2}\left(\overline{V}_{1\infty} + \overline{V}_{2\infty}\right)$ - is the asymptotic mean velocity;

$$\begin{aligned} H(z,\xi) &= \frac{1}{2ti} ctg \frac{\pi}{t}(z-\xi) \\ G(z,\xi) &= \frac{1}{2\pi i} \ln \sin \frac{\pi}{t}(z-\xi) \\ \hat{q}(\xi) &= 2\frac{\partial \overline{w}}{\partial \xi} = -\left(v_x \frac{\partial \ln p^*}{\partial \xi} - v_y \frac{\partial \ln p^*}{\partial \eta}\right), p^* = \frac{\varsigma \cdot h}{\varsigma_0}, \\ v_x &= \frac{\partial \varphi}{\partial x} = \frac{1}{p^*}\frac{\partial \psi}{\partial y}, \\ v_y &= \frac{\partial \varphi}{\partial y} = -\frac{1}{p^*}\frac{\partial \psi}{\partial x} \end{aligned} \tag{7}$$

$D^-_{0^*}$ - bounded simple convex domain, defined as: $D^-_{0^*} : \left[-\frac{t}{2} \langle \xi \langle \frac{t}{2}, -\left(t+\frac{l}{2}\right) \langle \eta \langle \left(t+\frac{l}{2}\right)\right]$ (8)

where *l* - is the projection of L_0 profile's frame on the Oy axis.

Based on the results of [5], [6] in the practical calculus of the complex conjugate velocity $\overline{w}(z)$, given by (6), the following iteration formula can be applied:

$$\overline{w}(z) = \overline{V}_m + \overline{w}_0^{(n)}(z) + \overline{w}_{\Delta q^*}^{(n-1)}, \quad n = 1,2,... \tag{9}$$

where: $\overline{w}_0^{(n)}(z) = \frac{1}{2it}\int_{L_0} \overline{w}^{(n)}(\xi) \cdot H(z,\xi)d\xi$

$$\begin{aligned} \overline{w}_{\Delta q^*}^{(n-1)} &= i\iint_{D^-_{0^*}} \Delta q^{*(n-1)} \cdot H(z,\xi)d\xi d\eta \\ \Delta q^{*(n-1)}(\xi) &= -\Delta v_x^{(n-1)} \frac{\partial \ln p^{*(n-1)}}{\partial \xi} - \Delta v_y^{(n-1)} \frac{\partial \ln p^{*(n-1)}}{\partial \eta} \\ p^{*(n-1)} &= \frac{(\varsigma \cdot h)^{(n-1)}}{\varsigma_0} \\ \varsigma^{(n-1)} &= \varsigma_0 \left(1 - \frac{k-1}{2}\frac{\left(\overline{w}^{(n-1)}\right)^2}{c_0^2}\right)^{\frac{1}{k-1}}, \text{ where } k \text{ is the adiabatic constant} \\ \left|\overline{w}^{(n-1)}\right|^2 &= \left(v_x^{(n-1)}\right)^2 + \left(v_y^{(n-1)}\right)^2 \\ c_0 &- \text{is the sound velocity} \end{aligned} \tag{10}$$

The successive approximation methods can be applied on (9) simultaneously over ς and *h*, or successively over ς and *h*. In the first approximation step it is assumed that $\varsigma = \varsigma_0 = const.$ and $h = const.$, hence $q^{*(0)}(\xi) = \Delta q^{*(0)}(\xi) = 0$, and (6) is solved without the double integral. For the velocity $\overline{w}^{(1)}(z) = v_x^{(1)} - iv_y^{(1)}$ thus obtained it is possible to determine $\varsigma^{(1)}, p^{*(1)}$. Next, $\Delta q^{*(l)}$ is calculated, and $\overline{w}_{\Delta q^*}^{(l)}$, is obtained from (10). Furthermore, we proceed similarly with the second approximation step, etc.

Another possibility for solving the fundamental equations (6) is given by the BEM method [6], i.e. solving the equations in real variables, using the results of [7]. For doing so, we consider the integral equation of the complex potential $F(z)=\varphi+i\psi$, and transform it into an integral equation with real variables, i.e. we build the integral equation of the velocity potential $\varphi(x,y)$.

❖ THE LAGRANGE INTERPOLATION POLYNOMIAL

The problem of constructing a continuously defined function from given discrete data is unavoidable whenever one wishes to manipulate the data in a way that requires information not included explicitly in the data. The relatively easiest and in many applications often most desired approach to solve the problem is *interpolation* [2], where an approximating function is constructed in such a way as to agree perfectly with the usually unknown original function at the given measurement points. In the practical application of the finite calculus of the problem of interpolation is the following: given the values of the function for a finite set of arguments, to determine the value of the function for some intermediate argument[2].

A chronological overview of the developments in interpolation theory, from the earliest times to the present date could be found in. In this section we focus our attention on the theory of the *lagrange interpolation polynomial* [2], since, as we have already mentioned in the proof of proposition 3.3, its usage arises also in our calculus algorithm for the study of the compressible fluid's stationary movement through profile grids on an axial-symmetric flow-surface in variable thickness of stratum.

The problem of interpolation consists in the following [2: Given the values yi corresponding to xi, i = 0, 1, 2, . . . , n, a function f(x) of the continuous variable x is to be determined which satisfies the equation:

$$y_i = f(x_i) \text{ for } i=0,1,2,...,n \text{ (11)} \tag{11}$$

and finally *f*(*x*) corresponding to $x=x'$ is required. (i.e. x' different from $x_i,\ i=\overline{1,n}$.)

In the absence of further knowledge as to the nature of the function this problem is, in the general case, indeterminate, since the values of the arguments other than those given can obviously assigned arbitrarily.

If, however, certain analytic properties of the function be given, it is often possible to assign limits to the error committed in calculating the function from values given for a limited set of arguments. For example, when the function is known to be representable by a polynomial of degree n, the value for any argument is completely determinate when the values for *n* + 1 distinct arguments are given.

Consider the function $f:[x_0,x_n]\to R$ given by the following table of values [2]:

x_k	x_0	x_1	...	x_n
$f(x_k)$	$f(x_0)$	$f(x_1)$	...	$f(x_n)$

x_k are called *interpolation nodes*, and they are not necessary equally distanced from each other. We seek to find a polynomial *P*(*x*) of degree n that approximates the function *f*(*x*) in the interpolation nodes, i.e.:

$$f(x_k)=P(x_k);k=0,1,2,...,n \tag{12}$$

The **Lagrange interpolation method** finds such a polynomial without solving the system (12).

Theorem 3.1. Lagrange Interpolating Polynomial

The Lagrange interpolating polynomial *is the polynomial of degree* n *that passes through* (n + 1) *points* $y_0=f(x_0), y_1=f(x_1),...y_n=f(x_n)$. *It is given by the relation ([2]):*

$$P(x)=\sum_{j=0}^{n}P_j(x) \tag{13}$$

where:

$$P_j(x)=y_j\prod_{k=0,k\neq j}^{n}\frac{x-x_k}{x_j-x_k} \tag{14}$$

Written explicitly:

$$\begin{aligned}P(x)=&\frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}y_0+\\&+\frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}y_1+\\&+\frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})}y_n\end{aligned} \tag{15}$$

For illustrating the usability of the Lagrange interpolation method through five points for our calculus algorithm for the study of the compressible fluid's stationary movement through profile grids on an axial-symmetric flow-surface in variable thickness of stratum, namely, for calculating the tangential velocity $v_\tau = \frac{d\varphi}{ds}$ (see section 3, proposition 3.3, equation (24)).

❖ SOLVING THE INTEGRAL EQUATION OF VELOCITY POTENTIAL

Our purpose is to solve the fundamental equations (6) (obtained from the CVBEM method) using (BEM) in real variables. For doing so, we consider the fundamental integral-equation of the complex potential $F(z) = \varphi + i\psi$ and transform it into an integral equation with real variables, i.e. we build the integral equation of the velocity potential $\varphi(s)$ ($\psi(s)$ is the flow rate function).

Theorem 3.2. *[6], [8] In the subsonic motion of the compressible fluid through the profile grid, on an axial-symmetric flow-surface, in variable thickness of stratum, the velocity potential* $\varphi(s)$, $s \in L_0$ *is the solution of the integral equation (16):*

$$\varphi(s) + \int_{L_0} \varphi(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b(s) + \iint_{D_{0^*}^-} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \tag{16}$$

where: s (x_0, y_0) and $\sigma(\xi,\eta)$ are the curvilinear coordinates of the fixed point A on the L_0 base profile;

$$b(s) = 2(x_0 V_{mx} + y_0 V_{my}) + \Gamma M(s,\sigma_A) + \int_{L_0} [\psi(s) - \psi(\sigma)] \frac{dN}{d\sigma} d\sigma$$

$$M(z_0,\zeta) = \frac{1}{\pi} arctg \frac{th \frac{\pi}{t}(\eta - y_0)}{tg \frac{\pi}{t}(\xi - x_0)} \tag{17}$$

$$N(z_0,\zeta) = \frac{1}{\pi} \ln \sqrt{\frac{1}{2}\left[ch \frac{2\pi}{t}(\eta - y_0) - \cos \frac{2\pi}{t}(\xi - x_0)\right]}$$

v_{mx}, v_{my} - are the components of the asymptotic mean velocity v_m.

Proposition 3.1. *[7],[8] In the case of an axial-subsonic movement of a perfect and compressible fluid through profile grids, the flow rate function is determined from the boundary condition (6):*

$$\psi(s) = u_0 \cdot \int_0^s p^*(s)\left(\frac{R}{R_0}\right) ds, \qquad u_0 = \omega R_0, \tag{18}$$

where: ω is the angular rotation velocity of the profile grid; R_0 defines the origin of the axis system related to the turbine's axis.

Equation (16) is an integro-differential equation. In this section, we will show a possibility of solving this equation applying the *method of successive approximation* (the iteration method), using also the result from [6] about the order of the term containing the double integral expression:

$$\varphi_{\hat{q}}(s) = \iint_{D_{0^*}^-} \hat{q}(\sigma) N(s,\sigma) d\xi d\eta \tag{19}$$

Proposition 3.2. *[6], [8] In the case of the subsonic movement of the compressible fluid through the profile grid on an axial-symmetric flow-surface, in variable thickness of stratum, the integral equation of the velocity potential* $\varphi : D_{0^*}^- \to \Re$ *is solvable by applying the method of successive approximations w.r.t. the parameter* $p^* = \frac{\varsigma \cdot h}{\varsigma_0}$.

Proof. For isentropic processes , by the Bernoulli-equation, we obtain:

$$\varsigma = \varsigma_0 \left(1 - \frac{\gamma - 1}{2} \frac{v^2}{c_0^2}\right)^{\frac{1}{\gamma - 1}}, v^2 = v_\tau^2 + v_n^2, v_\tau = \frac{d\varphi}{ds}, v_n = \frac{1}{p^*} \frac{d\psi}{ds} \tag{20}$$

where: γ is the adiabatic constant; c_0 is the sound velocity in the zero velocity point; v_τ and v_n are, respectively, the tangential and normal velocities on L_0.

In the first approximation it is assumed that $\varsigma = \varsigma_0$ = constant and $p^* = p^{*(0)}$ = constant. Thus, from (7), it results that $q^{(0)}(\sigma) = 0$. Hence, in the integral equation (4) the double integral (19) is neglected and results the following Fredholme integral equation of second type, with continuous nucleus:

$$\varphi^I(s) + \int_{L_0} \varphi^I(s) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b^I(s) \tag{21}$$

From solving equation (21) we obtain φ^I , and furthermore from (18), (20), (24) ψ^I, ς^I are obtained. Finally, using the relation:

$$p^* = \frac{\varsigma \cdot h}{\varsigma_0}, \qquad \hat{q}(\sigma) = -grad\varphi \cdot grad \ln p^* \tag{22}$$

a p^{*I} and $\hat{q}^I(\sigma)$ are determined.

In the second iteration $p^* = p^{*I}$ is assumed and for the determination of $\varphi^{II}(s)$ the following Fredholme integral equation of second type, with continuous nucleus, will be solved:

$$\varphi^{II}(s) + \int_{L_0} \varphi^{II}(\sigma) \frac{dM(s,\sigma)}{d\sigma} d\sigma = b^{II}(s) + \iint_{D^-_{0^*}} q^I(\sigma) N(s,\sigma) d\xi d\eta \tag{23}$$

where a φ^I and $b^{II}(s)$ are previously calculated from (18) and (17), respectively.

From solving equation (23), we obtain φ^{II} . Furthermore, from (18), (20), (24) and (22) ψ^{II}, ς^{II}, p^{*II} and $\hat{q}^{II}(\sigma)$ are obtained, respectively. Next, the third approximation might be done by assuming $p^*=p^{*II}$, and so on.

Proposition 3.3. *[7], [8] Having given the values of the velocity potential on each element of the* L_0 *profile's division, the tangential velocity* v_τ *may be calculated in each division element of the* L_0 *basic profile's boundary by the formula, given by the Lagrange interpolation method through five points:*

$$\begin{aligned} v_{\tau i} &= \frac{d\varphi}{ds}(s_i) = \frac{2}{3\Delta s_i}(\varphi_{i+2} - \varphi_{i-2}) - \frac{1}{12\Delta s_i}(\varphi_{i+4} - \varphi_{i-4}) \\ h &= \Delta s_i = s_{i+1} - s_{i-1}, \\ i &= 1,3,5,\ldots,2n-1, \end{aligned} \tag{24}$$

where n denotes the number of division elements and by s_i we refer to the i^{th} element of the division of L_0.

To ensure the practical functionality of proposition 3.2, i.e. to indicate the solving method of the Fredholme integral equation of second type obtained in each approximation step (equation (18), (23)), let us formulate and prove two more propositions.

Based on the results obtained by employing the interpolation formula through five points (24), we further determine the velocities $v_{\tau i}$ and with the size of the complex velocity w_i. Using such obtained values, in [8] we give an efficient algorithm for deriving the fluid's complex velocity through profile grids.

❖ CONCLUSION

We have shown some practical aspects of the usage of the calculus algorithm for the study of the compressible fluid's stationary movement through profile grids, on an axial-symmetric flow-surface, in variable thickness of stratum, namely:

- the usage of the boundary element method with real values;
- the applicability of the successive approximation method w.r.t. the parameters ς (fluid's density) and *h* (thickness variation of fluid stratum) for solving the integral equation of the velocity potential;
- the usage of the Lagrangian interpolation formula through five points for calculating the derivatives of the velocity potential.

Regarding practical applicability of our algorithm, our plans for the near future are:

- make more test cases w.r.t. several input (geometrical and hydrodynamical) values of the velocity potentials taken from practical experiments involving profile grids;
- study the possibility of applying the algorithm (i.e. the approximation methods) for the calculation of other fluid-characteristics.

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