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# VISCOUS FLOW OPTIMIZATION USING DESIGN OF EXPERIMENTS AND RESPONSE SURFACE METHOD

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**ABSTRACT:** *This paper aims in presenting considerations on experimental design and exposure of response surface method with applications in a matter of optimizing the flow of an incompressible viscous fluid through an obstructed cylindrical pipe. The results are pointed out, together with the advantages of using the response surface method in optimization problems.*
**KEYWORDS:** *design of experiment, response surface method, flow optimization*

## INTRODUCTION

*In a design problem or optimization, different solutions must be tested and compared. A trial or in our case a simulation is expensive (in computation time), so a systematic and orderly method is essential to solve the problems of design and optimization. Often engineering experimenters wish to find the conditions under which a certain process attains the optimal results. That is, they want to determine the levels of the design parameters at which the response reaches its optimum. The optimum could be either a maximum or a minimum of a function of the design parameters. One of methodologies for obtaining the optimum is response surface technique.*

*This paper is concerned with the method of response surface (RSM) as one approach in the optimization methodology.*

*Originally, RSM was developed to model experimental responses [1] and then migrated into the modeling of numerical experiments. The difference is in the type of error generated by the response. In physical experiments, inaccuracy can be due, for example, to measurement errors while, in computer experiments, numerical noise is a result of incomplete convergence of iterative processes, round-off errors or the discrete representation of continuous physical phenomena [3, 9, 10]. In RSM, the errors are assumed to be random.*

*The purpose of this research is to remove shortcomings by considering the classical method parameters and process responses as variables subject to random errors and the use of multiple regression and dispersion analysis of mathematical statistics to obtain the optimal parameters.*

*Combination of multiple regression, statistical analysis and programming dispersion of the experiences of modern methods of experimental research results, known as response surface method (regression functions), is applied in this work to optimize the flow of a viscous incompressible fluid through an obstructed duct.*

*The paper is outlined as follows: Section 2 presents the optimization problem and the mathematical model of the viscous flow through the obstructed duct. In Section 3 theoretical aspects upon the response surface method are exposed. The statistical design of experiments and the analysis of variance are presented in Section 4 and the numerical results and final remarks conclude the paper in Section 5.*

## OPTIMIZATION PROBLEM OVERVIEW

*It is considered a pipe of dimensions Lx, Ly, obstructed inside by a bar of square cross-section a, which is crossed by a viscous incompressible fluid (Figure 1). Due to the pipeline symmetry, only the upper part is considered as computational domain.*

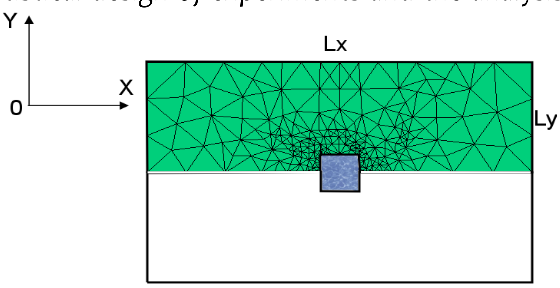


*Figure 1. Computational domain of the obstructed duct*

*The steady viscous flow of the incompressible fluid is governed by the Navier-Stokes equations in two-dimensional Cartesian coordinates*

$$\begin{cases} u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{\partial p}{\partial x}+\dfrac{1}{Re}\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right) \\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=-\dfrac{\partial p}{\partial y}+\dfrac{1}{Re}\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right) \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0 \end{cases} \tag{1a}$$

*associated with the boundary conditions*

$$v(x,Ly)=0,\quad u\left(\frac{Lx}{2}-\frac{a}{2},y\right)=0,\ y\in\left[0,\frac{a}{2}\right] \tag{1b}$$

$$p(0,y)=p_0,\quad p(Lx,y)=0, \tag{1c}$$

*where* $u(x,y)$, $v(x,y)$ *are the axial and cross velocity, respectively, Re is the Reynolds number defined by relation* $Re=\rho u d\eta^{-1}$, *where* $\rho$ *is the fluid density,* $\eta$ *is the kinematic viscosity coefficient and d is the duct diameter.*

*The stationary laminar flow it is governed by the Hagen-Poiseuille law which gives the relationship between volume of fluid flowing through a cylindrical tube of and pressure gradient*

$$Q_V=\frac{\pi(p1-p2)}{2\eta\ell}\int_0^R\left(R^2-r^2\right)dr \tag{2}$$

*Therefore, the viscous fluid flow volume is proportional with the pressure gradient* $(p1-p2)/Lx$. *We propose to determine the optimal sizes Lx and Ly of the pipeline, which maximizes the volume flow per time unit.*

## RESPONSE SURFACE METHODOLOGY

*Response surface methodology is a collection of mathematical and statistical techniques for building empirical models. The objective of the RSM is the optimization of a response (output variable) which is influenced by several independent variables (input variables), using a specific design of experiment. An experiment is a series of tests, called runs, in which changes are made in the input variables in order to identify the reasons for changes in the output response [1, 7].*

*Response surface method [4] considers the relationship between process parameters and responses that surface features of the multidimensional space of variables. In experiments conducted by this method, independent variables are varied simultaneously, taking a limited number of values considered in the experiment, called levels.*

*With this method, although several independent variables are varied simultaneously, their main effects and higher order, and interactions can be determined separately. Changing the independent variables will automatically lead to a change of output. The results thus obtained can be used to improve the performance of a manufacturing process.*

*The design procedure of response surface methodology involves three basic concepts:*

*(i) Designing of a series of experiments for adequate and reliable measurement of the response of interest.*

*(ii) Developing a mathematical model of the second order response surface with the best fittings.*

*(iii) Finding the optimal set of experimental parameters that produce a maximum or minimum value of response.*

*In general, the mathematical model of a process considers its response functional relationship between k parameters indicated by the physical reality of the process, as independent variables* $x_1,x_2,\ldots x_k$ *and its characteristic as a dependent variable or response* $\eta=f(x_1,x_2,\ldots x_k)$.

*We define the response y as*

$$y=\eta+\varepsilon_{exp}\ \text{ with } E(y)=\eta \text{ and } Variance(y)=\sigma^2 \tag{3}$$

where $\varepsilon_{exp}$ represents the noise or experimental error observed in the response, usually considered to be a random variable with zero mean and variance $\sigma^2$. Assuming that there is a deterministic relationship $f$ between $\eta$ and $x_1, x_2, \dots x_k$ we can write

$$y = f\left(x_1, x_2, \dots x_k\right) + \varepsilon_{exp} \tag{4}$$

The surface represented by $f\left(x_1, x_2, \dots x_k\right)$ is called a response surface.

Generally, the structure of the relationship between the response and the independent variables is unknown. The first step in RSM is to find a suitable approximation to the true relationship. The most common forms are low-order polynomials (first or second-order). Usually a second-order model is utilized in response surface methodology

$$y(x) = \beta_0 + \sum_{j=1}^{q} \beta_j x_j + \sum_{i<j}\sum \beta_{ij} x_i x_j + \sum_{j=1}^{q} \beta_{jj} x_j^2 + \varepsilon \tag{5}$$

where $\varepsilon$ is a random error. To determine the coefficients $\beta$ of the polynomial model, the most appropriate is the method of least squares [6], which provides a minimum dispersion of determined coefficients. The method of least squares, proposed by Gauss since the early nineteenth century, to adjust a function of regression of experimental data, is based on minimizing the sum of the squares of differences between variables $\eta_i$ of response measured in the experimental points and the corresponding y values calculated using the polynomial approximation determined

$$E = min \sum_{i=1}^{q} \left(y_i - \eta_i\right)^2 . \tag{6}$$

In general (5) can be written in matrix form

$$Y = BX + E \tag{7}$$

a matrix of independent variables. The matrices **B** and **E** consist of coefficients and errors, respectively. The solution of (7) can be estimated using the least-squares method as:

$$B = \left(X^T X\right)^{-1} X^T Y \tag{8}$$

where $X^T$ is the transpose of the matrix **X** and $\left(X^T X\right)^{-1}$ is the inverse of the matrix $X^T X$.

The mathematical models were evaluated for each response by means of multiple linear regression analysis. As said previous, modeling was started with a quadratic model including linear, squared and interaction terms. The significant terms in the model were found by analysis of variance (ANOVA) for each response. Significance was judged by determining the probability level that the F-statistic calculated from the data is less than 5%. The model adequacies were checked by $R^2$, adjusted-$R^2$, predicted-$R^2$ and prediction error sum of squares (PRESS). A good model will have a large predicted $R^2$ and a low PRESS. After model fitting was performed, residual analysis was conducted to validate the assumptions used in the ANOVA. This analysis included calculating case statistics to identify outliers and examining diagnostic plots such as normal probability plots and residual plots. Maximization and minimization of the polynomials thus fitted was usually performed by desirability function method and mapping of the fitted responses was achieved using computer software such as Matlab Statistical Toolbox.

## PIPE SIZE OPTIMIZATION BY RESPONSE SURFACE METHOD

Let us consider that the parameters to be optimized have values between ranges $Lx \in [3,5]$ and $Ly \in [1,2]$ respectively, and the cross-section of the block that obstructs the fluid is $a = 1.2$. The fluid element is a heat treatment oil having density $\rho = 0.897\ Kg/m^3$ and kinematic viscosity at 40° $\eta = 0.910\ Kg/m \cdot s$.

Using a central composite design [4] (Figure 2) value pairs were set to optimize parameters taken using extreme values and the ranges centers. These values and corresponding volume flow determined by simulations based on finite element method [5] are presented in Table 1.

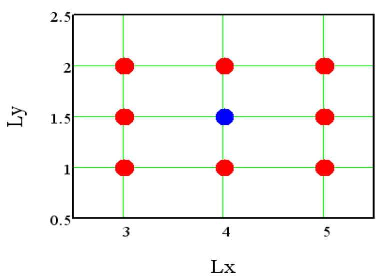


Figure 2. Central composite design used in response surface methodology, with factors Lx and Ly

Table 1. The values of parameters and values used in optimization of experimental responses

| Experiment order | Lx | Ly | Volume flow, $\eta$ |
|---|---|---|---|
| 1 | 3 | 1.5 | 1.49896 |
| 2 | 4 | 1.5 | 2.00022 |
| 3 | 5 | 1.0 | 2.49967 |
| 4 | 4 | 2.0 | 2.00343 |
| 5 | 3 | 1.0 | 1.49960 |
| 6 | 4 | 1.5 | 2.12546 |
| 7 | 3 | 2.0 | 1.49923 |
| 8 | 4 | 1.5 | 1.99875 |
| 9 | 4 | 1.0 | 1.99886 |
| 10 | 4 | 1.5 | 2.10120 |
| 11 | 4 | 1.5 | 1.89475 |
| 12 | 5 | 2.0 | 2.49706 |
| 13 | 5 | 1.5 | 2.49942 |

Table 2. Encoding of the experimental observations

| Observation | $x_1$ | $x_2$ | $\eta$ |
|---|---|---|---|
| 1 | $x_1^1$ | $x_2^1$ | $\eta^1$ |
| 2 | $x_1^2$ | $x_2^2$ | $\eta^2$ |
| ... | ... | ... | ... |
| $N=13$ | $x_1^N$ | $x_2^N$ | $\eta^N$ |

We consider the objective function

$$y=\beta_0+\beta_1x_1+\beta_2x_2+\beta_3x_1x_2+\beta_4x_1^2+\beta_5x_2^2+\varepsilon \qquad (9)$$

In matrix notation, relation (9) become

$$y=[x]^T[\beta]+\varepsilon \qquad (10)$$

where $[x]$ represents the vector of model parameters and their contributions

$$[x]^T=\begin{bmatrix}1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2\end{bmatrix} \qquad (11)$$

and $[\beta]$ is the regression coefficients vector

$$[\beta]=\begin{bmatrix}\beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5\end{bmatrix} \qquad (12)$$

Noting the corresponding observations as Table 2 and associating each observation a vector $[x]^i$, this design is characterized by

$$[X]=\begin{pmatrix}[x]^{1T}\\ [x]^{2T}\\ \dots\\ [x]^{NT}\end{pmatrix},\ [Y]=\begin{pmatrix}y_1\\ y_2\\ \dots\\ y_N\end{pmatrix},\ [E]=\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \dots\\ \varepsilon_N\end{pmatrix} \qquad (13)$$

thus applying equation (10) for all experimental observations, the algebraic system of equations in matrix form is obtained

$$[Y]=[X][\beta]+[E] \qquad (14)$$

Table 3. The estimated model coefficients

| | Term | Coef |
|---|---|---|
| $\beta_0$ | Constant | -0.316905 |
| $\beta_1$ | Lx | 0.613094 |
| $\beta_2$ | Ly | 0.149090 |
| $\beta_3$ | Lx*Lx | -0.0139613 |
| $\beta_4$ | Ly*Ly | -0.0480293 |
| $\beta_5$ | Lx*Ly | -0.00111700 |

It is noted that the system (14) results in more equations than unknowns, thus the error vector $[E]$ must be minimized. Using the least squares method, the coefficients vector $[\beta]$ is determined so the sum of the squared errors $\varepsilon_i$ is minimized, i.e.

$$\left\|[E]\right\|=min\left(\sum_{i=1}^{q=13}\varepsilon_i\right)^{1/2} \qquad (15)$$

The estimated coefficients $\left[\hat{\beta}\right]$ are obtained by relation

$$\left[\hat{\beta}\right]=\left([X]^T[X]\right)^{-1}[X]^T[Y] \qquad (16)$$

Model coefficients, determined by experimental design program Minitab are presented in Table 3.

To validate the regression model is necessary to calculate the $R^2$ - factor

$$R^2=\frac{SS_R}{SS_T} \qquad (17)$$

where the sum of squares of the residual is

$$SS_R=\sum_{i=1}^{N}\left(y_i-\eta_i\right)^2 \qquad (18)$$

and the total sum of squares

$$SS_T=\sum_{i=1}^{N}\left(y_i-\bar{\eta}\right)^2,\ \bar{\eta}=\frac{1}{N}\left(\eta_1+\eta_2+\dots+\eta_N\right) \qquad (19)$$

is a measure of the balance of all the observations.

For the regression model presented was calculated $R^2=97.75\%$, value that validates the model.

The response surface that was obtained is shown in Figure 3.

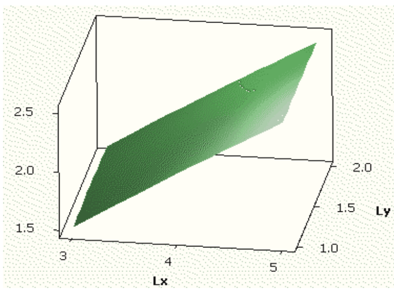


Figure 3. The response surface

We determine the coordinates (Lx, Ly) that maximizes the response surface using the constrained optimization Matlab code, listed in Appendix. The resulting optimal parameters are $Lx=4.8579,\quad Ly=1.4988$.

## NUMERICAL RESULTS AND CONCLUDING REMARKS

*The pipe dimensions were optimized using response surface method so that the flow volume passing the obstructed duct per unit time has a maximum value. The optimized flow is depicted in Figure 4. The axial and the cross-section velocity components are depicted in Figure 5. The variation of the wall pressure is captured in Figure 6.*

*Response surface method ensures the desired optimal model with no errors and also in a calculation time as small, independent of the hardware that is processing.*

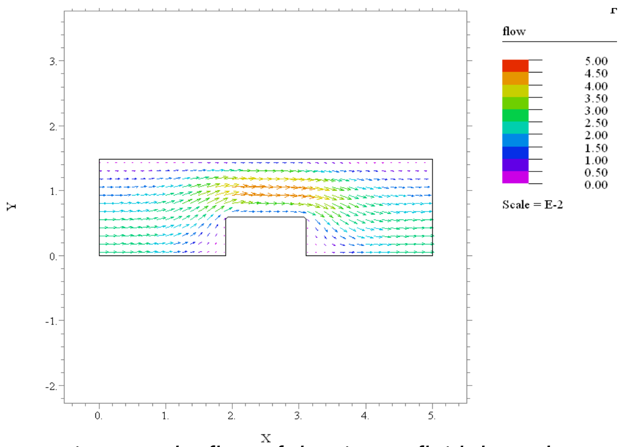


*Figure 4. The flow of the viscous fluid through the optimized duct*

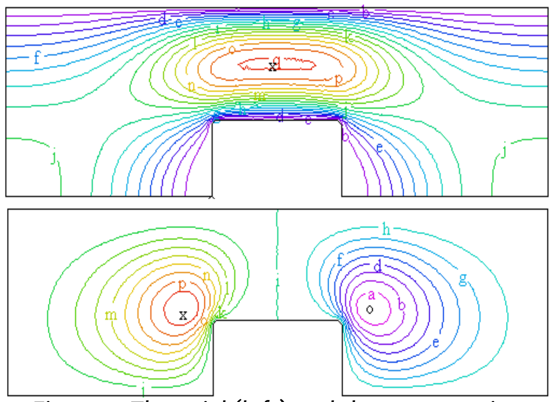

*Figure 5. The axial (left) and the cross-section (right) velocity components*

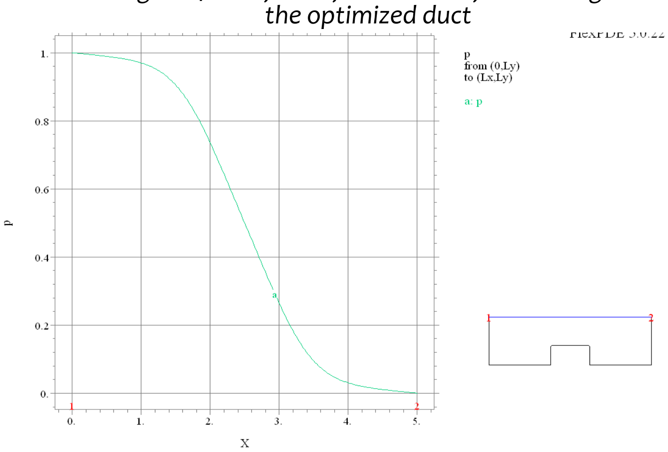


*Figure 6. The variation of the wall pressure*

*Using the results of numerical simulations the optimization of the size of a pipe traversed by a viscous obstructed fluid was performed by the response surface method.*

*The purpose of this paper was to remove the shortcomings of the classical optimization methods by considering parameters and process responses as variables subject to random errors and the use of multiple regression and dispersion analysis of mathematical statistics to obtain optimal parameters. Experimental design is currently the most modern tool used in optimization problems. This contributes to some important clarifications on the relationship between variables, estimation connections, checking assumptions, testing different ways of practical action, determining the optimal level of controlled variables and model behavior in relation to the variation factors.*

## APPENDIX

*We present in the following the code for bound constrained optimization used in the response surface design.*

*Step 1: Write a file objfun.m for the objective function.*

```
function f = objfun(x)
beta0=-0.316905;
beta1=0.613094;
beta2=0.149090;
beta11=-0.0139613;
beta22=-0.048293;
beta12=-0.001117;
f=beta0+beta1*x(1)+beta2*x(2)+beta11*x(1)^2+beta22*x(2)^2+beta12*x(1)*x(2);
```

*Step 2: Write a file confun.m for the constraints.*

```
function [c, ceq] = confun(x)
% Bounded constraints
c = [3-x(1);
  x(1)-5;
  1-x(2);
  x(2)-2];
% Nonlinear equality constraints
ceq = [];
```

*Step 3: Invoke constrained optimization routine.*

```
        x0 = [3,1];
        lb = [3,1];
        ub = [5,2];
        [x,fval] = fmincon(@objfun,x0,[],[],[],[],lb,ub,@confun)
x =
   4.8579    1.4988
fval =
   2.5067
```

## REFERENCES


[1.] Box, G.E.P., Draper, N.R., Empirical model-building and response surfaces, New York: John Wiley & Sons, 1987.

[2.] Conslik, J., Optimal Response Surface Design in Monte Carlo Sampling Experiments, Annals of Economic and Social Measurement, 3/3, pp. 463-473, 1974.

[3.] Giunta, A.A., Balabanov, V., Haim, D., Grossman, B., Mason, W.H., Watson, L.T., Haftka, R.T., Wing design for a high-speed civil transport using a design of experiments methodology, AIAA paper 96-4001-CP, Proceedings of 6th AIAA/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Bellevue WA, 1996.

[4.] JMP, Design of Experiments, SAS Institute Inc., Cary, NC, USA, ISBN 1-59047-816-9, 2005.

[5.] Maksay, Şt., Bistrian, D., Introducere în metoda elementelor finite, Editura Cermi Iaşi, ISBN 978-973-667-324-5, 2008.

[6.] Miller, S., The Method of Least Squares, Mathematics Department Brown University Providence, RI 02912.

[7.] Montgomery, D.C., Design and Analysis of Experiments, Wiley, New York. 5th Edition, 2000.

[8.] Myers, R., Montgomery, D., Anderson-Cook, C.M., Response Surface Methodology: Process and Product Optimization Using Designed Experiments, Wiley, ISBN: 978-0-470-17446-3, 2009.

[9.] Toropov, V.V., van Keulen, F., Markine, V.L., de Boer, H., Refinements in the multi-point approximation method to reduce the effects of noisy responses, 6th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Bellevue WA, Part 2, 1996.

[10.] Van Campen, D.H., Nagtegaal, R., Schoofs, A.J.G., Approximation methods in structural optimization using experimental designs for multiple responses, In: Eschenauer, H., Koski, J., Osyczka, A. (Ed.), Multicriteria Design Optimization, Springer-Verlag, Berlin, Heidelberg, New York, 1990.


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