ANNALS OF FACULTY ENGINEERING HUNEDOARA
– International Journal of Engineering
Tome XI (Year 2013) – FASCICULE 3 (ISSN 1584 – 2673)

[1.] Sanjeev SHARMA, [2.] Manoj SAHNI

CREEP ANALYSIS OF THIN ROTATING DISC HAVING VARIABLE THICKNESS AND VARIABLE DENSITY WITH EDGE LOADING

[1-2.] DEPARTMENT OF MATHEMATICS, JAYPEE INSTITUTE OF INFORMATION TECHNOLOGY, SECTOR 62, NOIDA-201307, INDIA

ABSTRACT: Creep stresses have been obtained for a thin rotating disc having variable thickness and variable density with edge load. Results obtained have been discussed and depicted graphically. It has been seen that a rotating disc whose density and thickness decreases radially with edge load is much safer for a design in comparison to a flat rotating disc having variable density. The deformation is much more significant for rotating disk with edge load than those of a rotating disc without edge load.
KEYWORDS: Creep, Transition, Transversely Isotropic, Rotating Disc

INTRODUCTION

Rotating disc play an important role in machine design. As a matter of fact the stresses in discs depend on the angular velocity with which they rotate. The problem of rotating discs was first treated in the early nineteenth century. Solutions of the isotropic discs including variable thickness, variable density and other cases can be found in most of standard creep textbooks [1, 4, 7]. Creep of thick-walled homogeneous and non -homogeneous cylinders under internal pressure has been analyzed. Transition theory of creep has been given by Seth in 1972 in which he has defined the measure concept in mechanics [6] and creep transition [7]. You et al. [14] has calculated elastic-plastic stresses for a disk with variable thickness and density. Singh et al. [13] calculated creep stresses and strains for thick walled cylinders under internal pressure and temperature. A rigid inclusion case using transition theory has been taken by Gupta et al. [3] in 2007. A large number of research problems [2, 8-12] had been solved using transition theory.

Therefore, rotating discs were considered to be one of the exhausted subjects in the field of solid mechanics. There are many applications of rotating discs such as in turbines, rotors, and computer disk drives. With all these applications and interest, there has been much research in this field. The analytical procedures presently available are restricted to problems with the simplest configurations, possessing constant material properties and thickness.

In this paper, we investigated the influence of density on the creep stresses in a rotating thin disc with variable thickness and edge loading by using Seth's transition theory [7]. The thickness h and density of the disc ρ are assumed to vary along the radius in the form

$$h = h_0\left(\frac{r}{b}\right)^{-k} \quad \text{and} \quad \rho = \rho_0\left(\frac{r}{b}\right)^{-m} \tag{1}$$

where h_0 and ρ_0 are thickness and density at $r = b$ respectively, k and m are thickness and density parameters respectively. Results obtained have been discussed numerically and depicted graphically.

GOVERNING EQUATIONS

Consider a thin circular disc of variable thickness and variable density having internal radius a and external radius b respectively. The disc is rotating with angular velocity ω of gradually increasing magnitude about an axes perpendicular to its plane and passing through the centre. The disc is thin so that it is effectively in a state of plane stress $(T_{zz} = 0)$ and the variation of the thickness is radial and symmetric with respect to the mid plane. In cylindrical polar co-ordinates the displacements are given by [9],

$$u = r\,(1-\beta); \quad v = 0 \quad and \quad w = d\,z \tag{2}$$

where β is a function of $r = \sqrt{x^2 + y^2}$ and d is a constant.

The finite components of strain are [7],

$$e_{rr}^A = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right],\ e_{\theta\theta}^A = \frac{1}{2}\left[1-\beta^2\right],\ e_{zz}^A = \frac{1}{2}\left[1-(1-d)^2\right],\ e_{r\theta}^A = e_{\theta z}^A = e_{zr}^A = 0 \quad (3)$$

The generalized components of strain are [6],

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right], e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right],\ e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right],\ e_{r\theta} = e_{\theta z} = e_{zr} = 0 \quad (4)$$

where n is the measure and $\beta' = \dfrac{d\beta}{dr}$.

Stress-strain relations for this problem are

$$T_{rr} = \frac{2\lambda\mu}{(\lambda+2\mu)}\left[e_{rr}+e_{\theta\theta}\right]+2\mu\, e_{rr}, T_{\theta\theta} = \frac{2\lambda\mu}{(\lambda+2\mu)}\left[e_{rr}+e_{\theta\theta}\right]+2\mu\, e_{\theta\theta} \quad T_{zz} = T_{zr} = T_{r\theta} = T_{\theta z} = 0 \quad (5)$$

Substituting equations (4) in equations (5), the non-zero stress components are,

$$T_{rr} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{(1-C)+(2-C)(P+1)^n\right\}\right], T_{\theta\theta} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{(2-C)+(1-C)(P+1)^n\right\}\right] \quad (6)$$

where $r\beta' = \beta P$ and $C = \dfrac{2\mu}{(\lambda+2\mu)}$.

Equations of equilibrium are all satisfied except

$$\frac{d}{dr}\left(h\, r\, T_{rr}\right) - h\, T_{\theta\theta} + \rho\,\omega^2 r^2\, h = 0 \quad (7)$$

Using equation (6) in equation (7), we get a non-linear differential equation in β as

$$\begin{aligned}(2-C)\, n\, P\,(P+1)^{n-1}\beta^{n+1}\frac{dP}{d\beta} &= \left(\frac{rh'}{h}\right)\left[3-2C-\beta^n\left\{(1-C)+(2-C)(P+1)^n\right\}\right]+\\ &\beta^n\left[1-(P+1)^n-nP\left\{(1-C)+(2-C)(P+1)^n\right\}\right]+\frac{n\rho\omega^2 r^2}{2\mu}\end{aligned} \quad (8)$$

The transitional points of β in equation (8) are $P \to -1$ and $P \to \pm\infty$.

The boundary conditions are

$$T_{rr} = 0 \text{ at } r = a \text{ and } T_{rr} = T_0 \text{ at } r = b \quad (9)$$

SOLUTION THROUGH THE PRINCIPAL STRESS DIFFERENCE

It has been shown [7, 9] that the asymptotic solution through the principal stress difference at the transition point $P \to -1$ leads to the creep state. We define the transition function R as,

$$R = T_{rr} - T_{\theta\theta} = \left(\frac{2\mu\,\beta^n}{n}\right)\left[1-(P+1)^n\right] \quad (10)$$

Taking the logarithmic differentiation of equation (10) w.r.t. 'r', we get

$$\frac{d}{dr}(\log R) = \frac{nP}{r} - \frac{\left[nP\beta(P+1)^{n-1}\beta^n\dfrac{dP}{d\beta}\right]}{r\beta^n\left[1-(P+1)^n\right]} \quad (11)$$

Substituting the value of $\dfrac{dP}{d\beta}$ from equation (8) in equation (11), we get

$$\frac{d}{dr}(\log R) = \frac{nP}{r} - \frac{\left[\begin{array}{l}\dfrac{rh'}{h}\left\{(3-2C)-\beta^n\left[(1-C)+(2-C)(P+1)^n\right]\right\}+\dfrac{n\rho r^2\omega^2}{2\mu}+\\ +\beta^n\left[1-(P+1)^n-nP\left\{(1-C)+(2-C)(P+1)^n\right\}\right]\end{array}\right]}{r\beta^n(2-C)\left[1-(P+1)^n\right]} \quad (12)$$

Asymptotic value of equation (12) as $P \to -1$ is

$$\frac{d}{dr}(\log R) = -\left[\frac{n(3-2C)+1}{r(2-C)}\right]+\frac{h'}{h}\left(\frac{1-C}{2-C}\right)-\frac{r^n}{D^n(2-C)}\left[\frac{h'}{h}(3-2C)+\frac{n\rho\, r\omega^2}{2\mu}\right] \quad (13)$$

where asymptotic value of β as $P \to -1$ is D/r, D being a constant.

Integrating equation (13) with respect to 'r', we get

$$R = A\, r^g h^v \exp f \quad (14)$$

where $g=-\left[\dfrac{n(3-2C)+1}{(2-C)}\right]$, $\nu=\left(\dfrac{1-C}{2-C}\right)$ is the Poisson's ratio, A is a constant of integration and $f=-\dfrac{1}{(2-C)D^n}\int\left[(3-2C)\dfrac{h'}{h}+\dfrac{n\rho r\omega^2}{2\mu}\right]r^n dr$.

From equation (10) and (14), we have

$$T_{rr}-T_{\theta\theta}=A\,r^g h^\nu \exp f \tag{15}$$

Substituting equation (15) in (7) and integrating, we get

$$hT_{rr}=B-A\int Fdr-\omega^2\int \rho\, r\, h\, dr \tag{16}$$

where B is a constant of integration and $F=r^{g-1}h^{\nu+1}\exp f$.

Using boundary conditions (9) in equation (16), we get

$$A=-\frac{\left\{\omega^2\int_a^b \rho\, rhdr+h_0T_0\right\}}{\int_a^b Fdr} \quad \text{and} \quad B=\omega^2\left[\int \rho\, rhdr\right]_{r=a}+A\left[\int Fdr\right]_{r=a}.$$

Now substituting value of A and B in equation (16), we get

$$T_{rr}=\frac{\left\{\omega^2\int_a^b \rho\, h\, rdr+h_0T_0\right\}}{h\int_a^b Fdr}\int_a^r Fdr-\frac{\omega^2}{h}\int_a^b \rho\, h\, r\, dr \tag{17}$$

From equations (17) and (15), we have

$$T_{\theta\theta}=T_{rr}+\frac{\left\{\omega^2\int_a^b \rho\, r\, h\, dr+h_0T_0\right\}}{\int_a^b Fdr}\left(\frac{rF}{h}\right) \tag{18}$$

Equations (17)-(18) give creep stresses for a thin rotating disc having variable thickness and variable density.

Now we introduce the following non-dimensional quantities

$$R=\frac{r}{b};\; R_0=\frac{a}{b};\; \Omega^2=\frac{\rho\omega^2b^2}{T_0};\; \sigma_r=\frac{T_{rr}}{T_0};\; \sigma_\theta=\frac{T_{\theta\theta}}{T_0};\; E_1=\frac{E}{T_0}$$

Using equations (1) in equations (17) - (18), we get the stresses in non-dimensional form as

$$\sigma_r=A_1\int_{R_0}^R bF_1dR-\Omega^2R^k\left[\frac{R^{2-k-m}-R_0^{2-k-m}}{2-k-m}\right];\quad 2-k-m\neq 0 \tag{19}$$

$$\sigma_\theta=\sigma_r+A_1R^{g-k(\nu+1)}\,b^g\exp f_1 \tag{20}$$

where $A_1=\dfrac{\left\{\Omega^2\left[\dfrac{1-R_0^{2-k-m}}{2-k-m}\right]+1\right\}R^k}{\int_{R_0}^1 bF_1dR}$, $F_1=R^{g-1-k(\nu+1)}\,b^{g-1}\exp f_1$ and $f_1=\dfrac{kb^n(3-2C)R^n}{n(2-C)D^n}-\dfrac{n(3-2C)\,\Omega^2b^nR^{2+n-m}}{E_1(2-C)^2(2+n-m)D^n}$.

For $2-k-m=0$, equations (19) - (20) becomes

$$\sigma_r=A_2\int_{R_0}^R bF_2\,dR-\Omega^2R^k\log\left(\frac{R}{R_0}\right) \tag{21}$$

$$\sigma_\theta=\sigma_r+A_2R^{g-k(\nu+1)}\,b^g\exp f_2 \tag{22}$$

where $A_2=\dfrac{\left(1-\Omega^2\log R_0\right)}{\int_{R_0}^1 bF_2dR}R^k$, $F_2=R^{g-1-k(\nu+1)}\,b^{g-1}\exp f_2$ and $f_2=\dfrac{k\,b^n(3-2C)R^n}{n\,(2-C)D^n}-\dfrac{n\,(3-2C)\,\Omega^2b^nR^{2+n-m}}{E_1(2-C)^2D^n(2+n-m)}$.

For a disc made of incompressible material $C \to 0\ \left(\nu \to 0.5\right)$, *stresses (19) - (20) becomes*

$$\sigma_r = A_3 \int_{R_0}^{R} bF_3\, dR - \Omega^2 R^k \left[\frac{R^{2-k-m} - R_0^{2-k-m}}{2-k-m}\right] \;;\; 2-k-m \neq 0 \tag{23}$$

$$\sigma_\theta = \sigma_r + A_3\, R^{-0.5\{3(n+k)+1\}}\, b^{-0.5(3n+1)} \exp f_3 \tag{24}$$

where $A_3 = \dfrac{\left\{\Omega^2\left[\dfrac{1-R_0^{2-k-m}}{2-k-m}\right]+1\right\}R^k}{\int_{R_0}^{1} bF_3\, dR}$, $F_3 = R^{-1.5(n+k+1)}\, b^{-1.5(n+1)} \exp f_3$ *and* $f_3 = \dfrac{3\,k\,b^n R^n}{2\,n\,D^n} - \dfrac{3n\,\Omega^2 b^n R^{2+n-m}}{4E_1(2+n-m)D^n}$.

For $2-k-m=0$, *stresses given by equations (21) - (22) becomes*

$$\sigma_r = A_4 \int_{R_0}^{R} bF_4\, dR - \Omega^2 R^k \log\left(\frac{R}{R_0}\right) \tag{25}$$

$$\sigma_\theta = \sigma_r + A_4\, R^{-0.5\{3(n+k)+1\}}\, b^{-0.5(3n+1)} \exp f_4 \tag{26}$$

where $A_4 = \dfrac{\left(1-\Omega^2 \log R_0\right)}{\int_{R_0}^{1} bF_4\, dR} R^k$, $F_4 = R^{-1.5(n+k+1)}\, b^{-1.5(n+1)} \exp f_4$ *and* $f_4 = \dfrac{3\,k\,b^n R^n}{2\,n\,D^n} - \dfrac{3\,n\,\Omega^2 b^n R^{2+n-m}}{4E_1 D^n(2+n-m)}$.

DISC HAVING CONSTANT THICKNESS (K=0) AND VARIABLE DENSITY WITH EDGE LOADING

For a flat disc rotating with angular speed Ω^2 *having variable density with edge loading, stresses given by equations (23) and (24) becomes*

$$\sigma_r = A_7 \int_{R_0}^{R} bF_7\, dR - \Omega^2 \left[\frac{R^{2-m} - R_0^{2-m}}{2-m}\right] \;;\; 2-m \neq 0 \tag{27}$$

$$\sigma_\theta = \sigma_r + A_7\, R^{-0.5(3n+1)}\, b^{-0.5(3n+1)} \exp f_7 \tag{28}$$

where $A_7 = \dfrac{\left\{\Omega^2\left[\dfrac{1-R_0^{2-m}}{2-m}\right]+1\right\}}{\int_{R_0}^{1} bF_7\, dR}$, $F_7 = R^{-1.5(n+1)}\, b^{-1.5(n+1)} \exp f_7$ *and*

$$f_7 = -\frac{3\,n\,\Omega^2 b^n R^{2+n-m}}{4E_1(2+n-m)D^n}.$$

For $2-m=0$, *stresses given by equations (25) - (26) become*

$$\sigma_r = A_8 \int_{R_0}^{R} bF_8 dR - \Omega^2 \log\left(\frac{R}{R_0}\right) \tag{29}$$

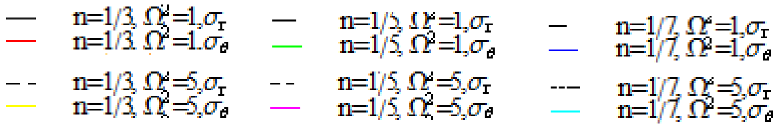
$$\sigma_\theta = \sigma_r + A_8 R^{-0.5(3n+1)} b^{-0.5(3n+1)} \exp f_8 \tag{30}$$

where $A_8 = \dfrac{1-\Omega^2 \log R_0}{\int_{R_0}^{1} bF_8 dR}$, $F_8 = R^{-1.5(n+1)} b^{-1.5(n+1)} \exp f_8$ *and* $f_8 = -\dfrac{3n\Omega^2 b^n R^{2+n-m}}{4E_1 D^n(2+n-m)}$

Results obtained are same as that of Gupta et.al. [4].

NUMERICAL ILLUSTRATION AND DISCUSSION

Curves have been drawn in figures 1 - 6 between stresses and radii ratio $R = r/b$ *for a disc rotating with angular speed* Ω^2 *= 1,5 having variable thickness (*k *= 0,1), variable density (*m *= -1,0,1) and* $n = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}$ $\left(i.e. N = 3,5,7\right)$ *with edge load* $E_1 = \frac{E}{T_0}$ *= 0.02, 1, 50.*



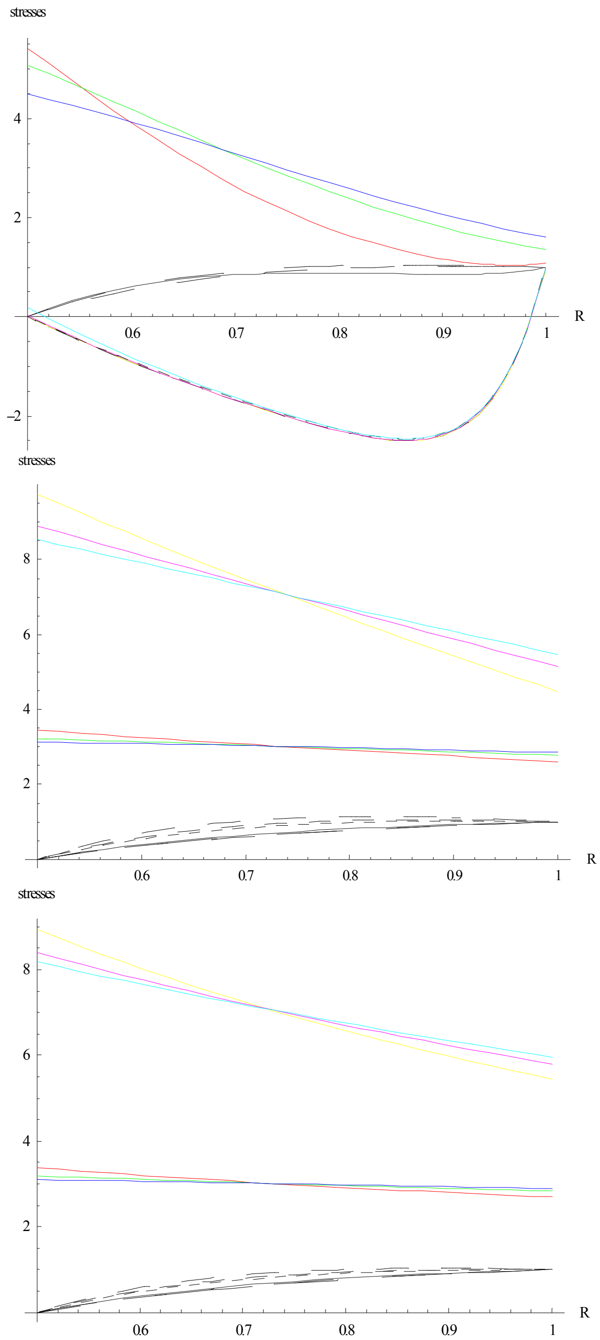


Figure 1: Creep stresses in a Thin Rotating Disc Having Variable Density (m = -1) and constant thickness (k = 0) for various radii ratios.

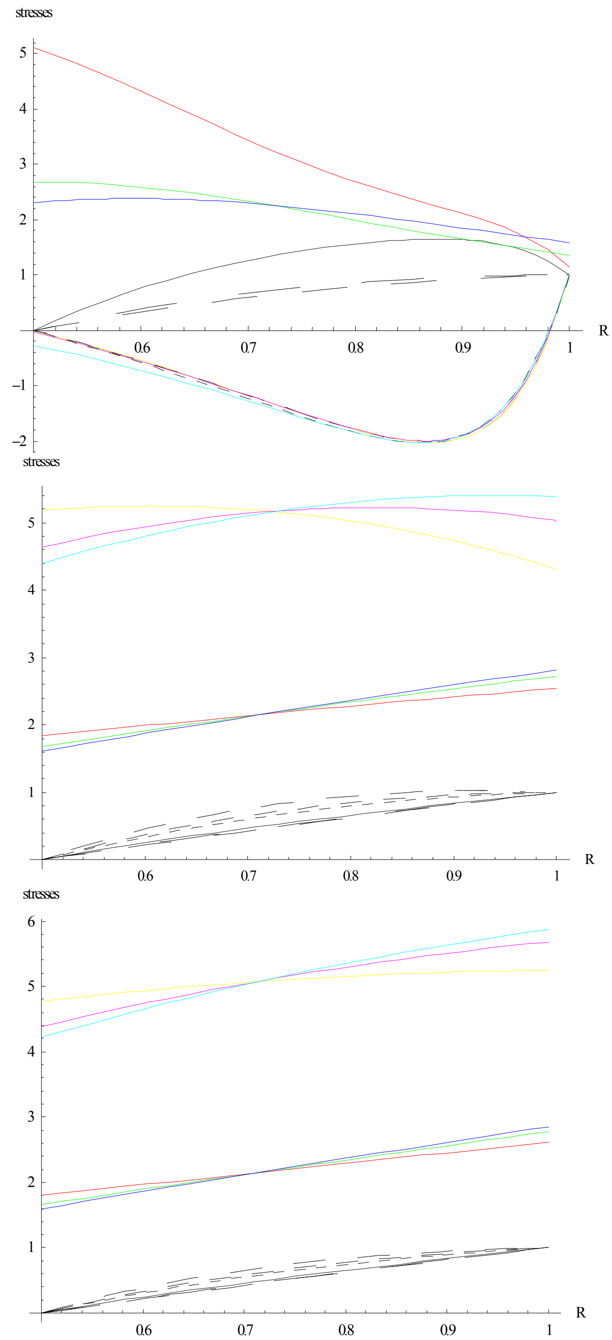


Figure 2: Creep stresses in a Thin Rotating Disc Having Variable Density (m = -1) and variable thickness (k = 1) for various radii ratios.

For edge load $E_1 < 1$, it is seen that for a flat disc ($k = 0$) whose density decreases radially ($m = 1$) and rotating with angular speed Ω^2 = 5, the circumferential stress is maximum at the internal surface for $n = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}$ respectively. For m = -1, 0, 1 ; k = 1 and $n = \frac{1}{7}$, though the circumferential stress is maximum at the internal surface yet it has smaller values than those for m = 1 and k = 0. This means that a flat disc with edge load ($E_1 < 1$) rotating with higher angular speed and whose density decreases radially increases the possibility of a fracture at the bore whereas a rotating disc having constant density (m = 0) or density increases radially (m = -1) and thickness decreases radially (k > 0) recedes the possibility of the fracture at the bore.

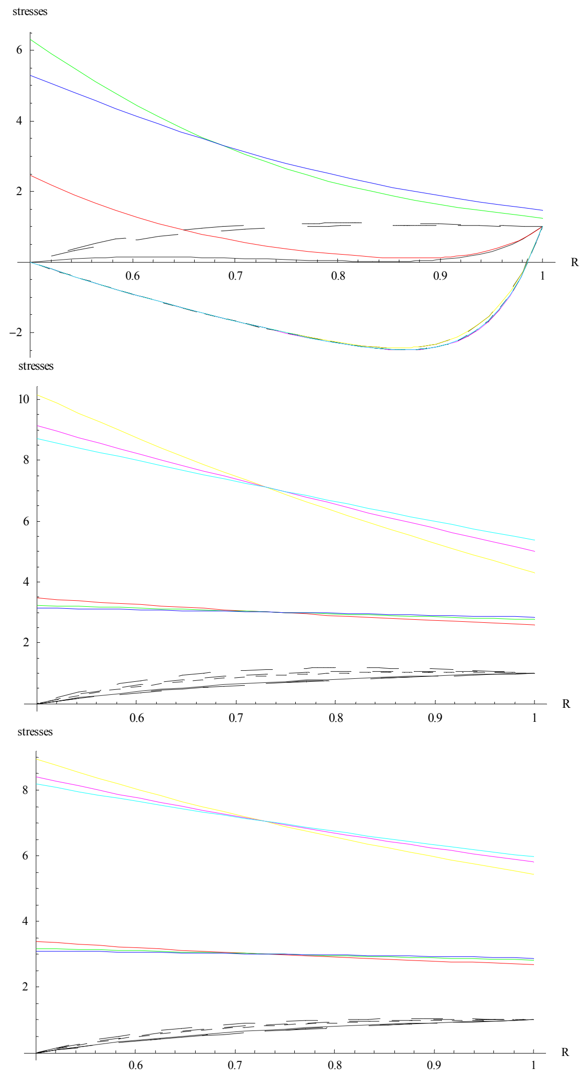


Figure 3: Creep stresses in a Thin Rotating Disc Having Constant Density (m = 0) and Constant thickness (k = 0) for various radii ratios.

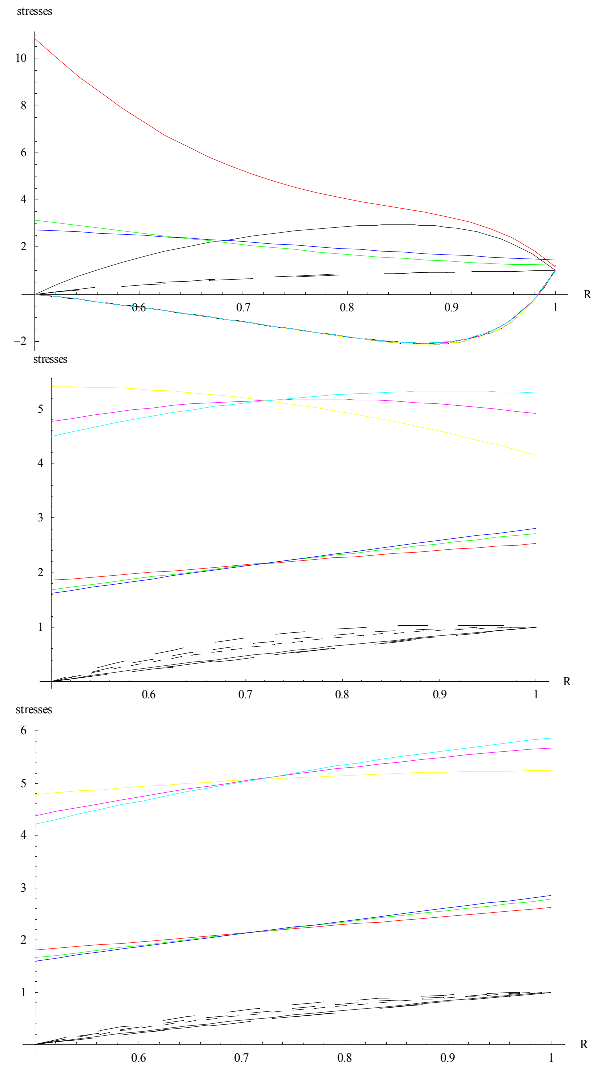


Figure 4: Creep stresses in a Thin Rotating Disc Having Constant Density (m = 0) and variable thickness (k = 1) for various radii ratios

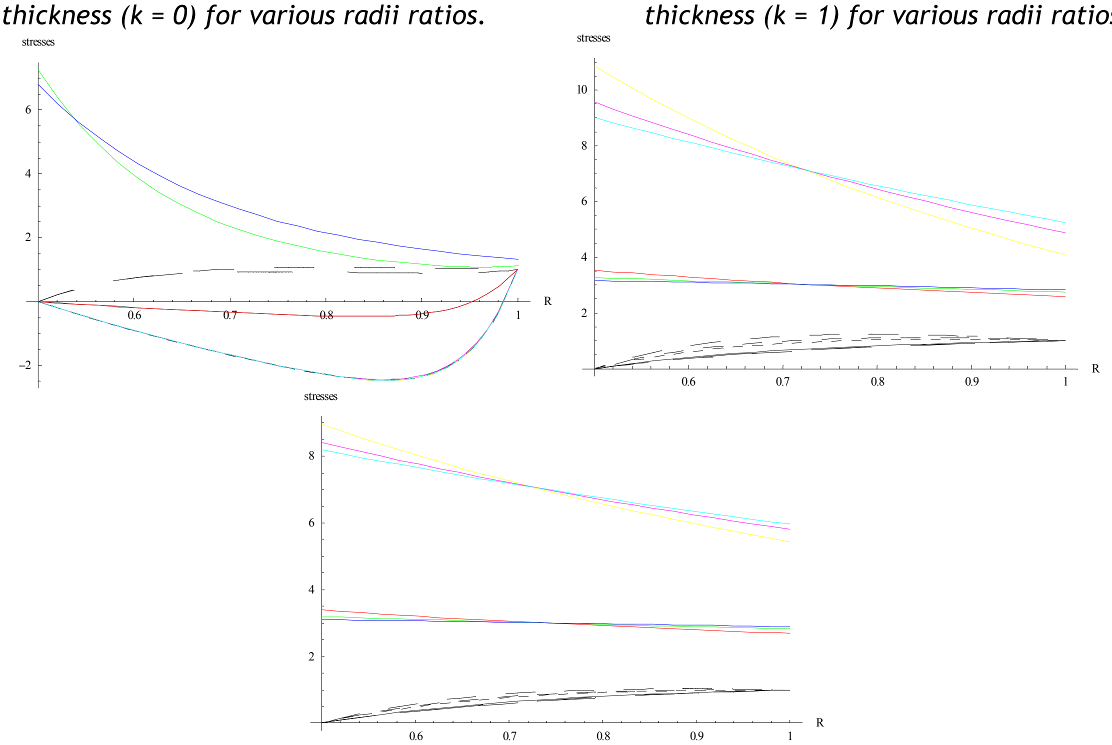


Figure 5: Creep stresses in a Thin Rotating Disc Having Variable Density (m = 1) and constant thickness (k = 0) for various radii ratios

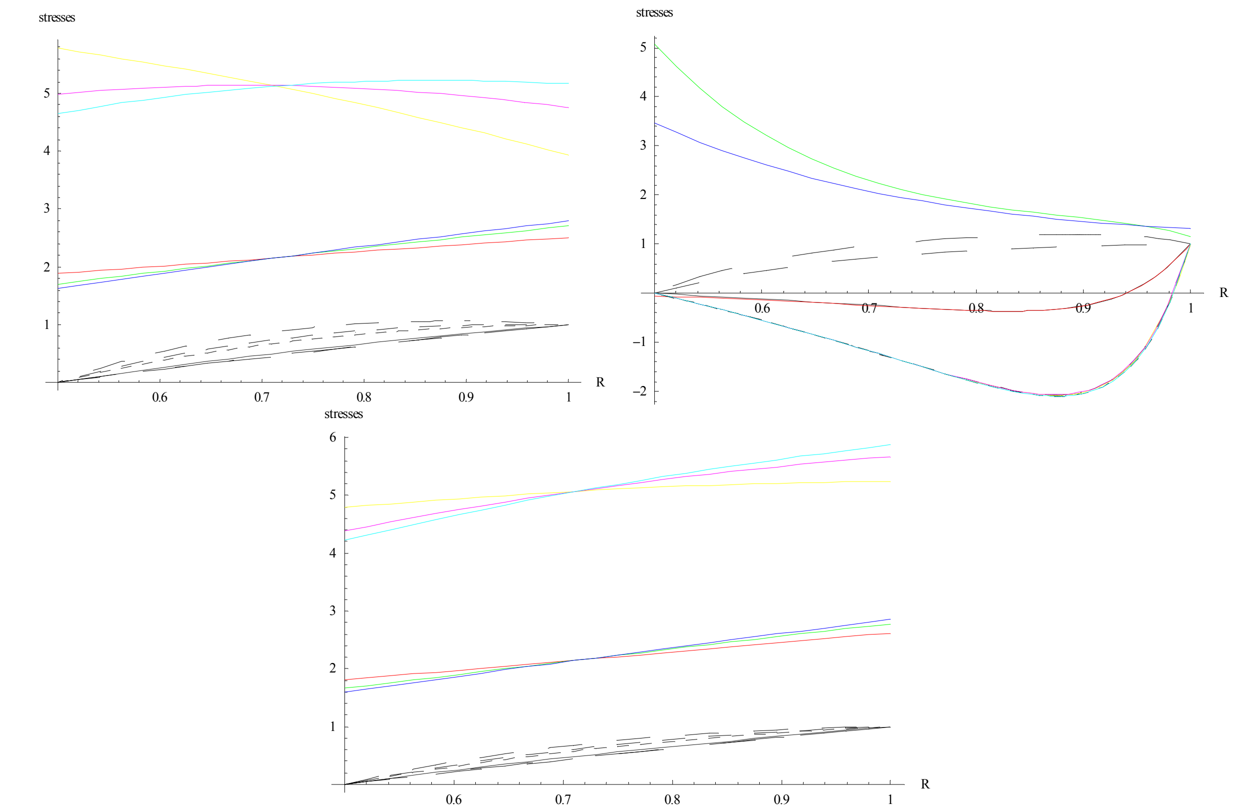


Figure 6: Creep stresses in a Thin Rotating Disc Having Variable Density (m = 1) and variable thickness (k = 1) for various radii ratios

For edge load $E_1 > 1$, variable density (m = -1, 0, 1), variable thickness (k = 1) and $n = \frac{1}{7}, \frac{1}{5}, \frac{1}{3}$; it is seen that the circumferential stress is maximum at the internal surface of a disc rotating with higher angular speed. For m = -1, 0 and k = 1 though the circumferential stress is maximum at the internal surface yet it has smaller values than those for m = 1 and k = 1.

Therefore, it can be concluded that a rotating disc whose density and thickness decreases radially with edge load $E_1 > 1$ and $n = \frac{1}{7}$ (or N = 7) is much safer for design in comparison to a flat rotating having variable density.

REFERENCES

[1] Boyle, J. T. and Spence J., Stress Analysis for Creep, Butterworths Company Ltd., Boston, 1983.

[2] Deepak, D., Gupta, V. K. and Dham, A. K., Creep Modeling in Functionally Graded Rotating Disc of Variable Thickness, Journal of Mechanical Science and Technology, 24(11), 2221 – 2232, 2010.

[3] Gupta, S. K. and Pankaj: Creep Transition in a Thin Rotating Disc with rigid inclusion, Def. Sc. Journal, 57(2), 185 – 195, 2007.

[4] Gupta, S. K., Sharma, Sanjeev and Pathak, Sonia, Creep Transition in a Thin Rotating Disc of Variable Density, Defence Science Journal, 50(2), 147-152, 2000.

[5] Kraus, H.: Creep Analysis, Wiley, New York, Toronto, 1980.

[6] Seth, B.R.: Measure Concept in Mechanics, Int. J. Non-linear Mech., I (2), 35 – 40, 1966.

[7] Seth, B. R.: Creep Transition, Jr. Math. Phys. Sci., 6, 1972.

[8] Sharma, S. and Sahni, M., Elastic-Plastic Transition of Transversely Isotropic Thin Rotating Disk, Contemporary Eng. Sci., 2(9), 433 – 440, 2009.

[9] Sharma, S., Sahai I. and Kumar, R., Creep Transition of a Thin Rotating Annular Disk of Exponentially Variable Thickness with Inclusion and Edge Load, Accepted-2013: Elsevier's Procedia Engineering.

[10] Sharma, S. and Sanehlata: Finite Difference Solution of Elastic-Plastic Thin Rotating Annular Disk with Exponentially Variable Thickness and Exponentially Variable Density, Hindawi's Journals of Materials, 1-9, 2013.

[11] Sharma, S., Thakur, P. and Sahni, M.: Elastic-plastic Analysis for Finite Deformation of a Rotating Disk of Exponentially Varying Thickness with Edge Load and Inclusion, Annals of Faculty Engineering Hunedoara- International Journal of Engineering, Tome X- Fascicule, 225-232, 2012.

[12] Sharma, S., Sahni, M. and Kumar, R.: Elastic Plastic Analysis of a Thin Rotating Disk of Exponentially Variable Thickness with Inclusion, WSEAS Transactions on Mathematics, 9(5), 315 – 323, 2010.

[13] Singh, T. and Gupta, V.K., Steady State Creep Analysis of a Functionally Graded Thick Cylinder Subjected to Internal Pressure and Thermal Gradient, International Journal of Material Research, 103(8), 1042 - 1051, 2012.

[14] You, L.H., Tang, Y.Y., Zhang, J.J. and Zheng C.Y.: Numerical analysis of elastic-plastic rotating disks with arbitrary variable thickness and density, International Journal of Solids and Structures, 37, 7809 -7820, 2000.