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ANALYSIS OF WATER FLOW IN AN UNSATURATED SOIL AS A POROUS MEDIA

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Abstract: In this study, a new reliable technique use to handle Richards' equation. This new method is resulted from VIM by a simple modification that is Reconstruction of Variational Iteration Method (RVIM). Unsaturated flow of soils in unsaturated soils is an important problem in geotechnical and geo-environmental engineering. Richards' equation is often used to model this phenomenon in porous media. Results are compared with those of homotopy perturbation method (HPM). The comparisons show that the Reconstruction of Variational Iteration Method is very effective and overcomes the difficulty of traditional methods and quite accurate to systems of non-linear partial differential equations.
Keywords: unsaturated soil; homotopy perturbation method; ariational iteration method; porous media

1. INTRODUCTION

The variational iteration method was proposed by He [1]. The variational iteration algorithm [2 and 3] in comparison with other analytical methods like ADM [4] and HPM [5] is of unheard simplicity, and the obtained results are of surprising accuracy (sometimes exact solutions can be obtained), the method can be easily understood by non-mathematical students and applied to various nonlinear problems. In addition, many authors gave great effort to give sophisticated theoretical verification of the variational iteration method, for example, Odibat [6], Salkuyeh [7], and Tatari & Dehghan [8] proved the variational iteration algorithm leads to convergent results. The method has been widely used to handle nonlinear models. The main property of the method is its flexibility and ability to solve nonlinear equations accurately and conveniently, for example, Wazwaz applied the method to various nonlinear systems including nonlinear wave equations and nonlinear diffusion equations, and concluded that the variational iteration method is a reliable analytical tool for solving nonlinear wave equations [9-11]; Goh, et al. obtained a similar conclusion that the variational iteration method is a reliable treatment for hyperchaotic systems [12]; Saberi-Nadjafi and Tamamgar pointed out that the method is a highly promising method for integro-differential equations [13]; Uremen and Yildirim obtained exact solutions of the Poisson equation [14] Sadighi also obtained exact solutions of nonlinear diffusion equations [15]; Altintan & Ugur [16], Hemeda [17] and Yusufoglu & Bekir [18] applied the method to, respectively, Sturm Liouville equation, wave equation, and the regularized long wave equation with great success.

In 2009 Hesameddini and Latifizadeh [19] proposed a new method based on Laplace transform - Reconstruction of variational iteration method (RVIM) in which the correctional function of the variational iteration method is obtained without using the Variational theory. Therefore in this method the complexity in calculating the Lagrange multiplier has been removed. In the present research, Reconstruction of variational iteration method (RVIM) has been employed to solve the problem of one-dimensional infiltration of water in unsaturated soil governed by Richards' equation. In the next sections, Richards' equation and the relative models involved are introduced, followed by a thorough explanation of the analytical method used to solve the equation. Illustrative examples are also given in order to demonstrate the effectiveness of the method in solving Richards' equation.

2. DESCRIPTION OF THE METHOD

In the following section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform [19] will be investigated a large of problems in science and engineering involve the solution of partial differential equations. Suppose x, t are two independent variables; consider t as the principal variable and x as the secondary variable. If $u(x,t)$ is a function of two variables x and t, when the Laplace transform is applied with t as a variable, definition of Laplace transform is

$$\mathbb{L}[u(x,t);s] = \int_0^\infty e^{-st} u(x,t)dt \tag{1}$$

We have some preliminary notations as

$$\mathbb{L}\left[\frac{\partial u}{\partial t};s\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial t}dt = sU(x,s) - u(x,0) \tag{2}$$

$$\mathbb{L}\left[\frac{\partial^2 u}{\partial t^2};s\right] = s^2U(x,s) - su(x,0) - u_t(x,0) \tag{3}$$

where

$$U(x,s) = \mathbb{L}[u(x,t);s] \tag{4}$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as $U(x,s),V(x,s)$ where $U(s)$ and $V(s)$ are known to the transform of the functions $u(x,t),v(x,t)$ respectively. The convolution of $u(x,t)$ and $v(x,t)$ is written $u(x,t)*v(x,t)$. It is defined as the integral of the product of the two functions after one is reversed and shifted.

Convolution Theorem: if $U(x,s),V(x,s)$ are the Laplace transform of $u(x,t),v(x,t)$, when the Laplace transform is applied to t as a variable, respectively; then $U(x,t).V(x,t)$ is the Laplace Transform of $\int_0^t u(x,t-\mathcal{E})v(x,\mathcal{E})d\mathcal{E}$

$$\mathbb{L}^{-1}[U(x,s).V(x,s)] = \int_0^t u(x,t-\mathcal{E})v(x,\mathcal{E})d\mathcal{E} \tag{5}$$

To facilitate our discussion of Reconstruction of Variational Iteration Method, introducing the new linear or nonlinear function $h(u(t,x)) = f(t,x) - N(u(t,x))$ and considering the new equation, rewrite $h(u(t,x)) = f(t,x) - N(u(t,x))$ as

$$L(u(t,x)) = h(t,x,u)) \tag{6}$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace Transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) \tag{7}$$

where $P(s)$ is polynomial with the degree of the highest order derivative of the selected linear operator.

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) = \mathbb{L}[h\{(x,t,u)\}] \tag{8}$$

$$U(x,s) = \frac{\mathbb{L}[h\{(x,t,u)\}]}{P(s)} \tag{9}$$

Suppose that $D(s) = \frac{1}{P(s)}$, and $\mathbb{L}[h\{(x,t,u)\}] = H(x,s)$, using the convolution theorem we have

$$U(x,s) = D(s).H(x,s) = \mathbb{L}\{(d(t) * h(x,t,u))\} \tag{10}$$

Taking the inverse Laplace transform on both side of Eq.

$$u(x,t) = \int_0^t d(t-\varepsilon)h(x,\varepsilon,u)d\varepsilon \quad (11)$$

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t d(t-\varepsilon)h(x,\varepsilon,u_n)d\varepsilon \quad (12)$$

And $u_0(x,t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x,t)$ should satisfy initial/ boundary conditions.

3. GOVERNING EQUATION AND APPLICATION OF METHOD

The basic theories describing fluid flow through porous media were first introduced by Buckingham [20] who realized that water flow in unsaturated soil is highly dependent on water content. Buckingham introduced the concept of "conductivity", dependent on water content, which is today known as unsaturated hydraulic conductivity [21]. This equation is usually known as Buckingham law [22]. Buckingham also went on to define moisture diffusivity which is the product of the unsaturated hydraulic conductivity and the slope of the soil-water characteristic curve. Nearly two decades later, Richards [23] applied the continuity equation to Buckingham's law – which itself is an extension of Darcy's law and obtained a general partial differential equation describing water flow in unsaturated, nonswelling soils with the matric potential as the single dependent variable [24]. There are generally three main forms of Richards' equation present in the literature namely the mixed formulation, the h-based formulation and the ɵ-based formulation, where h is the weight-based pressure potential and ɵ is the volumetric water content. Since Richards' equation is a general combination of Darcy's law and the continuity equation as previously mentioned, the two relations must first be written in order to derive Richards' equation.

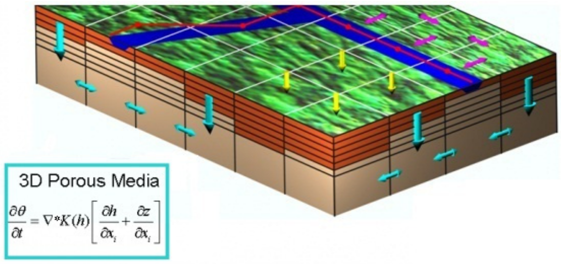

Figure 1. 3D Computer simulation for flow of water in porous media

As it can be seen, figure 1 shows the three-dimensional motion of surface water bodies, watersheds and groundwater which is presented by MOHID land. MOHID Water Modeling System is a physically-based, spatially distributed, continuous, variable time step model for the water and property cycles in inland waters. This program is designed to simulate hydrographic basin and aquifers (Figure 1) [25].

Here in, one-dimensional infiltration of water in vertical direction of unsaturated soil is considered and accordingly, Darcy's law and the continuity equation are given by Eqs. (13) and (14) respectively:

$$q = -k\frac{\partial H}{\partial Z} = -k\frac{\partial(h+z)}{\partial z} = -k\left(\frac{\partial h}{\partial Z}+1\right) \quad (13)$$

$$\frac{\partial\theta}{\partial t} = -\frac{\partial q}{\partial z} \quad (14)$$

where K is hydraulic conductivity, *H* is head equivalent of hydraulic potential, *q* is flux density and *t* is time. The mixed form of Richards' equation is obtained by substituting Eq. (13) in Eq. (14):

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[k\left(\frac{\partial h}{\partial z}+1\right)\right] \quad (15)$$

Equation (3) has two independent variables: the soil water content (ɵ) and pore water pressure head (h). Obtaining solutions to this equation therefore requires constitutive relations to describe the interdependence among pressure, saturation and hydraulic conductivity. However, it is possible to eliminate either ɵ or h by adopting the concept of differential water capacity, defined as the derivative of the soil water retention curve:

$$C(h) = \frac{d\theta}{dh} \quad (16)$$

The h-based formulation of Richards' equation is thus obtained by replacing Eq. (16) into Eq. (15):

$$C(h) = \frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial h}{\partial z}\right) + \frac{\partial k}{\partial z} \quad (17)$$

This is a fundamental equation in geotechnical engineering and is used fore modeling flow of water through unsaturated soils. For instance, the two-dimensional form of the equation can be used to model seepage in the unsaturated zone above water table in an earth dam. Introducing a new term D, pore water diffusivity defined as the ration of the hydraulic conductivity and the differential water capacity, the θ-based form of Richards' equation may be obtained. D can be written as:

$$D = \frac{K}{C} = \frac{K}{\frac{d\theta}{dh}} = K\frac{dh}{d\theta} \tag{18}$$

It should be noted that both *D* and *K* are highly dependent on water content. Combining Eq. (18) with Eq. (15) gives Richards' equation as:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial \theta}{\partial z}\right) + \frac{\partial K}{\partial z} \tag{19}$$

In order to solve Eq. (19), one must first properly address the task of estimating *D* and *K*, both of which are dependent on water content. Several models have been suggested for determining these parameters. The Van Genuchten model [26] and Brooks and Corey's model [27 and 28] are the more commonly used models. The Van Genuchten model uses mathematical relations to re15 late soil water pressure head with water content and unsaturated hydraulic conductivity, through a concept called "relative saturation rate". This model matches experimental data but its functional form is rather complicated and it is therefore difficult to implement it in most solution schemes. Brooks and Corey's model on the other hand has a more precise definition and is therefore adopted in the present research. This model uses the following relations to define hydraulic conductivity and water diffusivity:

$$D(\theta) = \frac{K_s}{a\lambda(\theta_s - \theta_r)}\left(\frac{\theta - \theta_r}{\theta - \theta_s}\right)^{2+\frac{1}{\lambda}} \tag{20}$$

$$K(\theta) = K_s\left(\frac{\theta \quad \theta_r}{\theta_s - \theta_r}\right)^{3+\frac{2}{\lambda}} \tag{21}$$

Where *Ks* is saturated conductivity, θ_r is residual water content, θ_s is saturated water content and a and λ are experimentally determined parameters. Brooks and Corey determined λ as pore-size distribution index. A soil with uniform pore-size possesses a large λ while a soil with varying pore-size has small value. Theoretically, the former can reach infinity and the latter can tend towards zero. Further manipulation of Brooks and Corey's model yields the following equations [27 and 28]:

$$D(\theta) = D_0(n+1)\theta^m \qquad m \geq 0 \tag{22}$$

$$K(\vartheta) = K_0\theta^k \qquad k \geq 1 \tag{23}$$

where K_0, D_0 and *k* are constants representing soil properties such as pore-size distribution, particle size, etc. In this representation of *D* and *K*, θ is scaled between 0 and 1 and diffusivity is normalized so that for all values of *m*, $\int D(\vartheta)d\theta = 1$ [29]. Several analytical and numerical solutions to Richards' equation exist based on Brooks and Corey's representation of *D* and *K*. Replacing *n*=0 and *k*=2 in Eqs. (22) and (23) yields the classic Burgers' equation extensively studied by many researchers [30-32]. The generalized Burgers' equation is also obtained for general values of *k* and *m* [33]. As seen previously, the two independent variables in Eq. (19) are time and depth. By applying the traveling wave technique [34 and 35] of time and depth, a new variable which is a linear combination of them is found. Tangent-hyperbolic function is commonly applied to solve these transform equations [36 and 37]. Therefore the general form of Burgers' equation in order of (*n*, 1) is obtained as:

$$\theta_t + a\theta^n\theta_z - \theta_{zz} = 0 \tag{24}$$

The exact solution to Eq. (24) can be found to be:

$$\theta(z,t) = \left(\frac{Y}{2} + \frac{Y}{2}\tanh\left(\lfloor A_1(z - A_2 t\rfloor)\right)\right)^{\frac{1}{n}}, \quad A_1 = \frac{-an+n|a|}{4(1+n)}Y \qquad (n \neq 0), \quad A_2 = \frac{Ya}{1+n} \tag{25}$$

In this study, nonlinear infiltration of water in unsaturated soil has been studied using Richards' equation and by employing Brooks and Corey's model to represent hydraulic conductivity and diffusivity. The conductivity term has been selected in two independent example cases as $\theta^2/2$ and $\theta^3/3$ and respective n-value of one was associated with these conductivities. Homotopy perturbation method (HPM) [38] and variational iteration method (VIM) have been used to solve Eq. (24). For solving Eq. (24) by means of RVIM, we choose the initial solution as follows,

$$\theta_0(x,t) = 0.5 - 0.5\tanh(0.25x) \tag{26}$$

By choosing linear term of θ_t, taking laplace transform and constructing iteration formula (12), we have;

$$\theta_1(x,t) = 0.5 - 0.5\tanh(0.25x) + 0.25\tanh(0.25x)\cdot(0.25 - 0.25\tanh(0.25x)^2)\cdot t \\ +(0.5 - 0.5\tanh(0.25x)(0.125 - 0.125\tanh(0.25x)^2)\cdot t \tag{27}$$

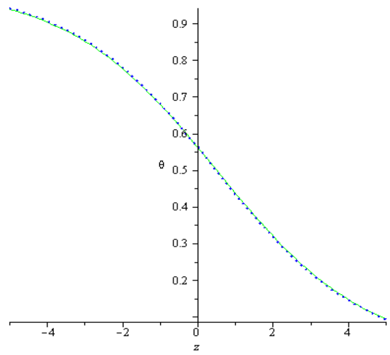

Figure 2. Plot of θ(z) for n=1.

And for $\theta_2(x,t), \theta_3(x,t), \ldots$, it can be calculated in the same way.

Results obtained through the first iteration of RVIM are compared with results of HPM in fifth order, Figure 2 – Blue line and black points represent results from first iteration of RVIM and fifth order of HPM, respectively.

Comparisons for second order of RVIM with fifth order of HPM are made in Figure 3 and Figure 4. Also Table 1 shows results of second order of RVIM, fifth order of HPM, third order of VIM with exact solution. It is obvious that the RVIM method is capable of solving such this equation as well.

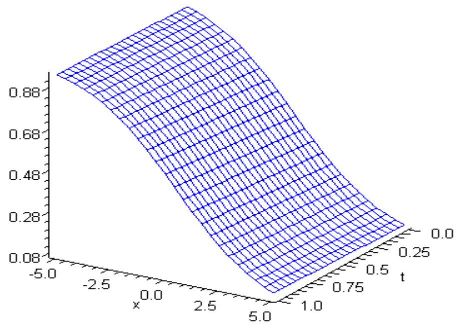

Figure 3. A 3D Computer simulation of RVIM results for n=1

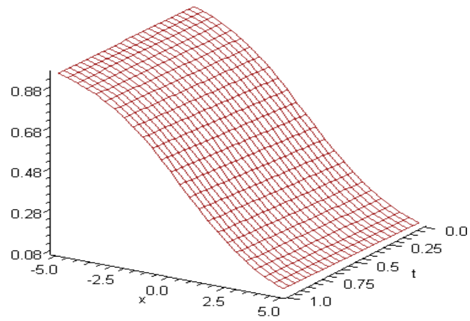

Figure 4. A 3D Computer simulation of HPM results for n=1.

Table 1. Comparing results of RVIM with given by HPM, VIM and Exact, when α=1 and n=1

t	0.5				1				2			
x	VIM	Exact	RVIM	HPM	VIM	Exact	RVIM	HPM	VIM	Exact	RVIM	HPM
- 3	0.872745	0.835470	0.835459	0.835483	0.926326	0.851849	0.851753	0.851952	1.02827	0.879452	0.879129	0.880796
- 1	0.710118	0.651360	0.651367	0.651354	0.796772	0.679201	0.679270	0.679178	0.966643	0.731421	0.731639	0.731067
1	0.466134	0.407348	0.407383	0.407333	0.555732	0.438051	0.438231	0.437823	0.738368	0.5030211	0.503364	0.499992
3	0.239113	0.201821	0.201827	0.201813	0.297392	0.222784	0.222819	0.222700	0.419163	0.269417	0.270017	0.268940

4. CONCLUSION

In this Letter, the Reconstruction of Variational Iteration Method (RVIM) was used for solving the Richard's equation. The obtained solutions are compared with Homotopy Perturbation method (HPM). Comparison of the RVIM with the HPM technique illustrates to us high accuracy, simplicity and speed of this method more than that of HPM and many of the results attained in this letter confirm the idea that RVIM is powerful mathematical tool for solving different kinds of practical problems. The reliability of this method and the reduction in the size of computational domain give this method a wider applicability in engineering.

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